Navier–Stokes Transport Coefficients of d-Dimensional Granular Binary Mixtures at Low Density

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Abstract The Navier–Stokes transport coefficients for binary mixtures of smooth inelastic hard disks or spheres under gravity are determined from the Boltzmann kinetic theory by application of the Chapman–Enskog method for states near the local homogeneous cooling state. It is shown that the Navier–Stokes transport coefficients are not affected by the presence of gravity. As in the elastic case, the transport coefficients of the mixture verify a set of coupled linear integral equations that are approximately solved by using the leading terms in a Sonine polynomial expansion. The results reported here extend previous calculations (Garzó, V., Dufty, J.W. in Phys. Fluids 14:1476–1490, 2002) to an arbitrary number of dimensions and provide explicit expressions for the seven Navier–Stokes transport coefficients in terms of the coefficients of restitution and the masses, composition, and sizes of the constituents of the mixture. In addition, to check the accuracy of our theory, the inelastic Boltzmann equation is also numerically solved by means of the direct simulation Monte Carlo method to evaluate the diffusion and shear viscosity coefficients for hard disks. The comparison shows a good agreement over a wide range of values of the coefficients of restitution and the parameters of the mixture (masses and sizes).

Keywords Granular binary mixtures · Inelastic Boltzmann equation · Navier–Stokes transport coefficients · DSMC method

1 Introduction

The simplest model for a granular fluid is a system composed by smooth hard spheres or disks with inelastic collisions. The only difference from the corresponding model for normal

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fluids is the loss of energy in each binary collision, characterized by a (constant) coefficient of normal restitution. For a low density gas, the Boltzmann kinetic equation conveniently modified to account for inelastic collisions [1–3] has been used in recent years as the starting point to derive the hydrodynamic-like equations of the system. Thus, assuming the existence of a *normal* (hydrodynamic) solution for sufficiently long space and time scales, the Chapman–Enskog (CE) method [4] has been applied to solve the Boltzmann equation to Navier–Stokes (NS) order and get explicit expressions for the transport coefficients. While this goal has been widely covered in the case of a monocomponent gas [5–7], much less is known for systems composed by grains of different masses, sizes, and concentrations (granular mixtures).

Needless to say, the determination of the NS transport coefficients for a multicomponent granular fluid is much more complicated than for a single granular system. Many attempts to derive these coefficients [8–11] have been carried out by means of the CE expansion around Maxwellians at the *same* temperature T for each species. The use of this distribution can only be considered as acceptable for nearly elastic systems where the assumption of the equipartition of energy still holds. In addition, according to this level of approximation, the inelasticity is only accounted for by the presence of a sink term in the energy balance equation and so the expressions of the NS transport coefficients are the same as those obtained for elastic collisions. However, as the dissipation increases, different species of a granular mixture have different partial temperatures T_i and consequently, the energy equipartition is seriously broken ($T_i \neq T$). The failure of energy equipartition in granular fluids [12, 13] has also been confirmed by computer simulations [14–22] and even observed in real experiments [23, 24] of agitated mixtures. All the results show that deviations from equipartition depend on the mechanical differences between the particles of each species and the coefficients of restitution of the system. Given that the inclusion of nonequipartition effects increases the level of complexity of the problem, it is interesting from a practical point of view to assess the influence of this effect on the transport properties of the system. If the NS transport coefficients turned out to be quite sensitive to nonequipartition, the predictions made from previous theories [8–11] should be reexamined by theories that take into account the nonequipartition of energy. For this reason, although the possibility of nonequipartition was already pointed out many years ago [25], a careful study of its influence on transport has only been carried out recently. In this context, Garzó and Dufty [26] have developed a kinetic theory for a binary granular mixture of inelastic hard spheres at low density which accounts for nonequipartition effects. Their results show that in general the consequences of the temperature differences for the transport coefficients are quite significant, especially for strong dissipation [27, 28]. It is important to remark that the expressions derived in Ref. [26] for the Navier–Stokes transport coefficients do not limit their application to weak inelasticity. In fact, the results reported in this paper include a domain of both weak and strong inelasticity, $0.5 \leq \alpha \leq 1$, where α is the (common) coefficient of restitution considered. The accuracy of these theoretical predictions (based on a Sonine polynomial expansion) has been confirmed by Monte Carlo simulations of the inelastic Boltzmann equation in the cases of the diffusion coefficient [29] and the shear viscosity coefficient of a mixture heated by an external thermostat [30]. Exceptions to this good agreement are extreme mass or size ratios and strong dissipation, although these discrepancies are mitigated in part if one retains more terms in the Sonine polynomial expansion [29]. For small dissipation, the results derived by Garzó and Dufty [26] agree with those recently obtained by Serero et al. [31] in the first order of the order parameter $\epsilon_{ij} = 1 - \alpha_{ij}^2$.

The CE method solves the Boltzmann equation by expanding the distribution function of each species $f_i(\mathbf{r}, \mathbf{v}, t)$ around the *local* homogeneous cooling state (HCS) [12]. This state

plays the same role for granular gases as the local equilibrium distribution for a gas with elastic collisions. Given that the form of the distribution function $f_i^{(0)}$ of the HCS is not exactly known, one usually considers the first correction to a Maxwellian at the temperature for that species, namely, a polynomial in velocity of degree four (leading Sonine correction). However, the results derived for hard spheres clearly show that the influence of these non-Gaussian contributions to the transport coefficients are in general negligible, except in the case of the heat flux for quite large values of dissipation [28]. Accordingly, a theory incorporating the contributions coming from the deviations of the HCS from its Gaussian form does not seem necessary in practice for computing the NS transport coefficients of the mixture.

The objective of this paper is twofold. First, given that the results reported in Ref. [26] are limited to hard spheres, we extend here this derivation to an arbitrary number of dimensions d. This goal is not only academic since, from a practical standpoint, many of the experiments reported for flowing granular materials have created (quasi) *two-dimensional* systems by confining grains between vertical or on a horizontal or tilted surface, enabling data collection by high-speed video [32–37]. Regarding computer simulations, most of them consider hard disks to save computer time and memory. For these reasons, it would be desirable to provide experimentalists and simulators with theoretical tools to work when studying problems both in two and three dimensions. In addition, apart from its practical interest, it is also interesting from a fundamental view to explore what is the influence of dimensionality on the dependence of the transport coefficients on dissipation. As a second target, we want also to present a simplified theory with explicit expressions for the transport coefficients. As the algebra involved in the calculations of Ref. [26] is complex, the constitutive relations for the fluxes were not explicitly displayed in this paper. Although the work carried out here involves complex algebra as well, the use of Maxwellians at different temperatures for the distribution functions of each species in the reference state allows us to explicitly obtain expressions for the *seven* relevant transport coefficients of the mixture in terms of the mechanical parameters of the system: masses, sizes, composition and coefficients of restitution. To assess the degree of accuracy of our (approximated) expressions, we have also performed Monte Carlo simulations for the diffusion and the shear viscosity coefficients for hard disks ($d = 2$). As shown below, the good agreement found between the results derived in this paper with computer simulations justifies this simplification and allows one to obtain more simplified forms of the transport coefficients.

The plan of the paper is as follows. In Sect. 2, the inelastic Boltzmann equation and the corresponding hydrodynamic equations are recalled. The CE expansion adapted to the inelastic binary mixtures is formulated in Sect. 3. Assuming that gradients and dissipation are independent parameters, the Boltzmann equation is solved by means of an expansion in powers of the spatial gradients around the local HCS distribution $f_i^{(0)}$. It is shown that the use of the local HCS as the reference state is not an assumption of the CE method but a consequence of the exact solution to the Boltzmann equation in the zeroth-order approximation. Section 4 deals with the expressions for the NS transport coefficients. As in the case of elastic collisions, these coefficients are the solutions of a set of coupled linear integral equations which involves the (unknown) distributions $f_i^{(0)}$. The integral equations are solved by considering two approximations: First, $f_i^{(0)}$ is replaced by its Maxwellian form at the temperature T_i, and second, only the leading terms in a Sonine polynomial expansion of the first-order distribution $f_i^{(1)}$ are retained. Technical details of the calculations carried out here are given in Appendices 1, 2 and 3. A comparison with previous results [31] based on the use of Maxwellians at the same temperature T as a ground state is also illustrated

in Sect. 4, showing significant discrepancies between both descriptions at moderate dissipation. Section 5 is devoted to the numerical solutions of the Boltzmann equation by using the direct simulation Monte Carlo (DSMC) method [38] in the cases of the diffusion D and shear viscosity η coefficients for hard disks. To the best of our knowledge, this is the first time that the NS shear viscosity of a granular binary mixture at low density has been numerically obtained from the DSMC method. The paper is closed in Sect. 6 with a brief discussion of the results presented in this paper.

2 Boltzmann Equation and Conservation Laws

Consider a binary mixture composed by smooth inelastic disks ($d = 2$) or spheres ($d = 3$) of masses m_1 and m_2, and diameters σ_1 and σ_2. The inelasticity of collisions among all pairs is characterized by three independent constant coefficients of restitution α_{11}, α_{22}, and $\alpha_{12} = \alpha_{21}$, where $\alpha_{ij} \leq 1$ is the coefficient of restitution for collisions between particles of species i and j. The mixture is in presence of the gravitational field so that each particle feels the action of the force $\mathbf{F}_i = m_i \mathbf{g}$, where $\mathbf{g}$ is the gravity acceleration. In the low density regime, the distribution functions $f_i(\mathbf{r}, \mathbf{v}; t)$ ($i = 1, 2$) for the two species are determined from the set of nonlinear Boltzmann equations [2]

$$\left(\partial_t + \mathbf{v} \cdot \nabla + \mathbf{g} \cdot \frac{\partial}{\partial \mathbf{v}} \right) f_i(\mathbf{r}, \mathbf{v}, t) = \sum_{j=1}^{2} J_{ij}[\mathbf{v}|f_i(t), f_j(t)], \tag{2.1}$$

where the Boltzmann collision operator $J_{ij}[\mathbf{v}|f_i, f_j]$ is

$$J_{ij}[\mathbf{v}_1|f_i, f_j] = \sigma_{ij}^{d-1} \int d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}} \, \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})$$
$$\times [\alpha_{ij}^{-2} f_i(\mathbf{r}, \mathbf{v}_1'', t) f_j(\mathbf{r}, \mathbf{v}_2', t) - f_i(\mathbf{r}, \mathbf{v}_1, t) f_j(\mathbf{r}, \mathbf{v}_2, t)]. \tag{2.2}$$

In (2.2), d is the dimensionality of the system, $\sigma_{ij} = (\sigma_i + \sigma_j)/2$, $\hat{\boldsymbol{\sigma}}$ is an unit vector along the line of centers, Θ is the Heaviside step function, and $\mathbf{g}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity. The primes on the velocities denote the initial values $\{\mathbf{v}_1', \mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary (restituting) collision:

$$\mathbf{v}_1' = \mathbf{v}_1 - \mu_{ji}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\hat{\boldsymbol{\sigma}},$$
$$\mathbf{v}_2' = \mathbf{v}_2 + \mu_{ij}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\hat{\boldsymbol{\sigma}}, \tag{2.3}$$

where $\mu_{ij} \equiv m_i/(m_i + m_j)$. The relevant hydrodynamic fields are the number densities n_i, the flow velocity $\mathbf{u}$, and the temperature T. They are defined in terms of moments of the distributions f_i as

$$n_i = \int d\mathbf{v} f_i(\mathbf{v}), \qquad \rho \mathbf{u} = \sum_{i=1}^{2} m_i \int d\mathbf{v} \mathbf{v} f_i(\mathbf{v}), \tag{2.4}$$

$$nT = p = \sum_{i=1}^{2} n_i T_i = \sum_{i=1}^{2} \frac{m_i}{d} \int d\mathbf{v} V^2 f_i(\mathbf{v}), \tag{2.5}$$

where $\mathbf{V} = \mathbf{v} - \mathbf{u}$ is the peculiar velocity, $n = n_1 + n_2$ is the total number density, $\rho = m_1 n_1 + m_2 n_2$ is the total mass density, and p is the pressure. Furthermore, the third equality of (2.5) defines the kinetic temperatures T_i for each species, which measure their mean kinetic energies.

The collision operators conserve the particle number of each species and the total momentum but the total energy is not conserved:

$$\int d\mathbf{v} J_{ij}[\mathbf{v}|f_i, f_j] = 0, \tag{2.6}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} m_i \int d\mathbf{v} \mathbf{v} J_{ij}[\mathbf{v}|f_i, f_j] = 0, \tag{2.7}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} m_i \int d\mathbf{v} V^2 J_{ij}[\mathbf{v}|f_i, f_j] = -dnT\zeta, \tag{2.8}$$

where ζ is identified as the total "cooling rate" due to inelastic collisions among all species. At a kinetic level, it is also convenient to introduce the "cooling rates" ζ_i for the partial temperatures T_i. They are defined as

$$\zeta_i = \sum_{j=1}^{2} \zeta_{ij} = -\frac{m_i}{dn_i T_i} \sum_{j=1}^{2} \int d\mathbf{v} V^2 J_{ij}[\mathbf{v}|f_i, f_j], \tag{2.9}$$

where the second equality defines the quantities ζ_{ij}. The total cooling rate ζ can be written in terms of the partial cooling rates ζ_i as

$$\zeta = T^{-1} \sum_{i=1}^{2} x_i T_i \zeta_i, \tag{2.10}$$

where $x_i = n_i/n$ is the mole fraction of species i.

From (2.4–2.8), the macroscopic balance equations for the mixture can be obtained. They are given by

$$D_t n_i + n_i \nabla \cdot \mathbf{u} + \frac{\nabla \cdot \mathbf{j}_i}{m_i} = 0, \tag{2.11}$$

$$D_t \mathbf{u} + \rho^{-1} \nabla \cdot \mathsf{P} = \mathbf{g}, \tag{2.12}$$

$$D_t T - \frac{T}{n} \sum_{i=1}^{2} \frac{\nabla \cdot \mathbf{j}_i}{m_i} + \frac{2}{dn}(\nabla \cdot \mathbf{q} + \mathsf{P} : \nabla \mathbf{u}) = -\zeta T. \tag{2.13}$$

In the above equations, $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the material derivative,

$$\mathbf{j}_i = m_i \int d\mathbf{v} \mathbf{V} f_i(\mathbf{v}) \tag{2.14}$$

is the mass flux for species i relative to the local flow,

$$\mathsf{P} = \sum_{i=1}^{2} m_i \int d\mathbf{v} \mathbf{V}\mathbf{V} f_i(\mathbf{v}) \tag{2.15}$$

is the total pressure tensor, and

$$\mathbf{q} = \sum_{i=1}^{2} \frac{m_i}{2} \int d\mathbf{v} V^2 \mathbf{V} f_i(\mathbf{v}) \tag{2.16}$$

is the total heat flux.

The macroscopic balance equations (2.11–2.13) are not entirely expressed in terms of the hydrodynamic fields, due to the presence of the cooling rate ζ, the mass flux $\mathbf{j}_i$, the heat flux $\mathbf{q}$, and the pressure tensor P which are given as functionals of the distributions f_i. However, it these distributions can be expressed as functionals of the hydrodynamic fields, then the cooling rate and the fluxes also will become functional of the hydrodynamic fields through (2.9) and (2.14–2.16). Such expressions are called *constitutive* relations and they provide a link between the exact balance equations and a closed set of equations for the hydrodynamic fields. This hydrodynamic description can be derived by looking for a *normal* solution to the Boltzmann kinetic equation. A normal solution is one whose all space and time dependence of the distribution function f_i occurs through a functional dependence on the hydrodynamic fields,

$$f_i(\mathbf{r}, \mathbf{v}, t) = f_i[\mathbf{v}|x_1(\mathbf{r}, t), p(\mathbf{r}, t), T(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t)]. \tag{2.17}$$

As in previous works [26, 28], we have taken the set $\{x_1, p, T, \mathbf{u}\}$ as the $d + 3$ independent fields of the two-component mixture. These are the most accessible fields from an experimental point of view. The determination of this normal solution from the Boltzmann equation (2.1) is a very difficult task in general, unless the spatial gradients are small. In this case, the CE method gives an approximate solution.

3 Chapman–Enskog Solution

The CE method is a procedure to construct an approximate normal solution. It is perturbative, using the spatial gradients as the small expansion parameter. More specifically, one assumes that the spatial variations of the hydrodynamic fields n_i, $\mathbf{u}$, p, and T are small on the scale of the mean free path. For ordinary gases this can be controlled by the initial or boundary conditions. It is more complicated for granular gases, since in some cases (e.g., steady states such as the simple shear flow problem [39]) the boundary conditions imply a relationship between dissipation and some hydrodynamic gradient. As a consequence, there are examples for which the NS approximation is restricted to the quasi-elastic limit [39]. Here, we also assume that the spatial gradients are independent of the coefficients of restitution so that, the corresponding NS order hydrodynamic equations apply for small gradients but they are not limited *a priori* to weak inelasticity. It must be emphasized that our perturbation scheme differs from the one recently carried out by Serero et al. [31] where the CE solution is given in powers of both the hydrodynamic gradients (or equivalently, the Knudsen number) and the degree of dissipation $\epsilon_{ij} = 1 - \alpha_{ij}^2$. The results provided in Ref. [31] only agree with our results in the quasielastic domain (small ϵ_{ij}). Moreover, in the presence

of an external force it is necessary to characterize the magnitude of the force relative to gradients as well. As in the elastic case [4], it is assumed here that the magnitude of the gravity field is at least of first order in perturbation expansion.

For small spatial variations, the functional dependence (2.17) can be made local in space through an expansion in gradients of the hydrodynamic fields. To generate it, f_i is written as a series expansion in a formal parameter δ measuring the nonuniformity of the system,

$$f_i = f_i^{(0)} + \delta f_i^{(1)} + \delta^2 f_i^{(2)} + \cdots, \tag{3.1}$$

where each factor of δ means an implicit gradient of a hydrodynamic field. The local reference states $f_i^{(0)}$ are chosen such that they verify (2.4) and (2.5), or equivalently, the remainder of the expansion must obey the orthogonality conditions

$$\int d\mathbf{v}[f_i(\mathbf{v}) - f_i^{(0)}(\mathbf{v})] = 0, \tag{3.2}$$

$$\sum_{i=1}^{2} m_i \int d\mathbf{v}\mathbf{v}[f_i(\mathbf{v}) - f_i^{(0)}(\mathbf{v})] = \mathbf{0}, \tag{3.3}$$

$$\sum_{i=1}^{2} \frac{m_i}{2} \int d\mathbf{v} V^2[f_i(\mathbf{v}) - f_i^{(0)}(\mathbf{v})] = 0. \tag{3.4}$$

The time derivatives of the fields are also expanded as $\partial_t = \partial_t^{(0)} + \delta \partial_t^{(1)} + \cdots$. The action of the operators $\partial_t^{(k)}$ can be obtained from the balance equations (2.11–2.13) when one takes into account the corresponding expansions for the fluxes and the cooling rate. This is the usual CE method [4] for solving kinetic equations. The main difference in the case of inelastic collisions is that the reference state has a time dependence associated with the cooling that is not proportional to the gradients. As a consequence, terms from the time derivative $\partial_t^{(0)}$ are not zero. In addition, the different approximations $f_i^{(k)}$ are well-defined functions of the coefficients of restitution α_{ij}, regardless of the applicability of the corresponding hydrodynamic equations truncated at that order.

3.1 Zeroth-Order Approximation

To zeroth order in the gradients, (2.1) becomes

$$\partial_t^{(0)} f_i^{(0)} = \sum_{j=1}^{2} J_{ij}[f_i^{(0)}, f_j^{(0)}], \tag{3.5}$$

where use has been made of the fact that gravity is assumed to be of first order in the uniformity parameter δ. The balance equations to this order give

$$\partial_t^{(0)} x_1 = \partial_t^{(0)} u_\ell = 0, \qquad T^{-1}\partial_t^{(0)} T = p^{-1}\partial_t^{(0)} p = -\zeta^{(0)}, \tag{3.6}$$

where $\zeta^{(0)}$ is determined by (2.9) and (2.10) to zeroth order in the gradients. Since $f_i^{(0)}$ is a normal solution, then the time derivative in (3.5) can be written as

$$\partial_t^{(0)} f_i^{(0)} = -\zeta^{(0)}(T\partial_T + p\partial_p) f_i^{(0)} = \frac{1}{2}\zeta^{(0)} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_i^{(0)}). \tag{3.7}$$

The second equality in (3.7) follows from dimensional analysis which requires that the dependence of $f_i^{(0)}$ on p and T is of the form

$$f_i^{(0)}(V) = x_i \frac{p}{T} v_0^{-d} \Phi_i(V/v_0), \tag{3.8}$$

where $v_0(t) = \sqrt{2T(m_1 + m_2)/m_1 m_2}$ is a thermal velocity defined in terms of the temperature $T(t)$ of the mixture and Φ_i is a dimensionless function of the reduced velocity V/v_0. The dependence of $f_i^{(0)}$ on the magnitude of $\mathbf{V}$ follows from the isotropy of the zeroth-order distribution with respect to the peculiar velocity. Thus, the Boltzmann equation at this order reads

$$\frac{1}{2} \zeta^{(0)} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_i^{(0)}) \sum_{j=1}^{2} J_{ij}[f_i^{(0)}, f_j^{(0)}]. \tag{3.9}$$

Equation (3.9) has the same form as the Boltzmann equation for a strictly *homogeneous* state. The latter is called the homogeneous cooling state (HCS) [12]. Here, however, the state is not homogeneous because of the requirements (3.2–3.4). Instead it is a *local* HCS. It must be emphasized that the presence of this local HCS as the ground or reference state is not an assumption of the CE expansion but rather a consequence of the kinetic equations at zeroth order in the gradient expansion.

The local HCS distribution is the solution of the Boltzmann equation (3.9). However, its explicit form is not exactly known even in the one-component case [40]. An accurate approximation for the zeroth-order solution $f_i^{(0)}$ can be obtained by using low order truncation of a Sonine polynomial expansion. The results show that in general, $f_i^{(0)}$ is close to a Maxwellian at the temperature for that species. Further details of this solution for hard spheres ($d = 3$) can be found in Ref. [12]. An important consequence is that the kinetic temperatures of each species are different for inelastic collisions and, consequently the total energy is not equally distributed between both species (breakdown of energy equipartition). This violation of energy equipartition has been confirmed by computer simulation studies [14] as well as by real experiments [23]. The condition that $f_i^{(0)}$ is *normal* in the sense of (2.17) (namely, it depends on time only through its functional dependence on T and p) implies that the ratio $T_i/T \equiv \gamma_i(x_1)$ depends on the hydrodynamic state through the concentration x_1.

The dependence of the temperature ratio $\gamma \equiv \gamma_1/\gamma_2 = T_1/T_2$ on the parameters of the mixture is obtained by requiring that the partial cooling rates $\zeta_i^{(0)}$ must be equal [12], i.e.,

$$\zeta_1^{(0)} = \zeta_2^{(0)} = \zeta^{(0)}. \tag{3.10}$$

These partial cooling rates are nonlinear functionals of the distributions $f_i^{(0)}$, which are not exactly known. However, to get the temperature ratio, they can be well estimated by using Maxwellians at different temperatures:

$$f_i^{(0)}(\mathbf{V}) \to f_{i,M}(\mathbf{V}) = n_i \left(\frac{m_i}{2\pi T_i} \right)^{d/2} \exp\left(-\frac{m_i V^2}{2T_i} \right). \tag{3.11}$$

In this approximation, one gets [41]

$$\zeta_i^{(0)} = \sum_{j=1}^{2} \zeta_{ij}^{(0)} = \frac{4\pi^{(d-1)/2}}{d\Gamma(\frac{d}{2})} v_0 \sum_{j=1}^{2} n_j \mu_{ji} \sigma_{ij}^{d-1} \left(\frac{\theta_i + \theta_j}{\theta_i \theta_j} \right)^{1/2}$$

$$\times (1+\alpha_{ij})\left[1 - \frac{\mu_{ji}}{2}(1+\alpha_{ij})\frac{\theta_i + \theta_j}{\theta_j}\right], \qquad (3.12)$$

where

$$\theta_i = \frac{m_i}{\gamma_i}\sum_{j=1}^{2}m_j^{-1}. \qquad (3.13)$$

It must be remarked that the fact that $T_1(t) \neq T_2(t)$ does not mean that there are additional hydrodynamic degrees of freedom since the partial temperatures T_i can be expressed in terms of the granular temperature T as

$$T_1(t) = \frac{\gamma}{1 + x_1(\gamma - 1)}T(t), \qquad T_2(t) = \frac{1}{1 + x_1(\gamma - 1)}T(t). \qquad (3.14)$$

Note that the reference Maxwellians (3.11) for the two species can be quite different due to the temperature differences. This contrasts with the ground state considered in more standard derivations [8–11, 31] where $f_i^{(0)}$ is replaced by a Maxwellian defined at the same temperature T, i.e.,

$$f_i^{(0)}(\mathbf{V}) \rightarrow n_i\left(\frac{m_i}{2\pi T}\right)^{d/2}\exp\left(-\frac{m_i V^2}{2T}\right). \qquad (3.15)$$

As will show later, the approaches (3.11) and (3.15) lead to different results for the NS transport coefficients.

The solution to (3.10) gives the temperature ratio T_1/T_2 for any dimension d as a function of the mole fraction x_1, the mass ratio $\mu \equiv m_1/m_2$, the size ratio $\omega \equiv \sigma_1/\sigma_2$, and the coefficients of restitution α_{ij}. To illustrate the violation of equipartition theorem, in Fig. 1 we plot the temperature ratio versus the coefficient of restitution α for hard disks ($d = 2$) in the case of an equimolar mixture $x_1 = \frac{1}{2}$ for two different mixtures composed by particles of the same material: $\mu = 4$, $\omega = 2$, and $\mu = 8$, $\omega = \sqrt{8}$. For the sake of simplicity, we have taken a common coefficient of restitution $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$. We also include the simulation data obtained by solving numerically the Boltzmann equation by means of the DSMC method [38]. The excellent agreement between theory and simulation shows the accuracy of the estimate (3.12) to compute the temperature ratio from the equality of cooling rates (3.10). We also observe that the deviations from the energy equipartition increase as the mechanical differences between the particles of each species increase.

Fig. 1 Temperature ratio $\gamma \equiv T_1/T_2$ versus the (common) coefficient of restitution α for an equimolar mixture ($x_1 = 1/2$) of hard disks ($d = 2$) with $\omega = \mu^{1/2}$. Two different values of the mass ratio are considered: $\mu = 4$ and $\mu = 8$. The symbols refer to DSMC results while the lines represent the theoretical results obtained from the condition (3.10)
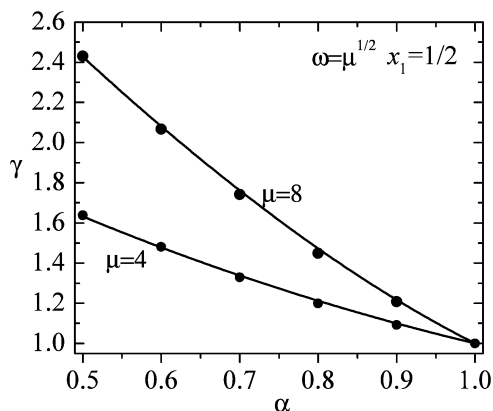
4 Navier–Stokes Transport Coefficients

The CE procedure allows one to determine the NS transport coefficients of the mixture in the first order of the expansion. The analysis to first order in gradients is similar to the one worked out in Ref. [26] for $d = 3$. Here, we only present the final results with some technical details being given in Appendix 1. The mass, momentum, and heat fluxes are given, respectively, by

$$\mathbf{j}_1^{(1)} = -\frac{m_1 m_2 n}{\rho} D\nabla x_1 - \frac{\rho}{p} D_p \nabla p - \frac{\rho}{T} D' \nabla T, \qquad \mathbf{j}_2^{(1)} = -\mathbf{j}_1^{(1)}, \tag{4.1}$$

$$P_{k\ell}^{(1)} = p\delta_{k\ell} - \eta\left(\nabla_\ell u_k + \nabla_k u_\ell - \frac{2}{d}\delta_{k\ell}\nabla\cdot\mathbf{u}\right), \tag{4.2}$$

$$\mathbf{q}^{(1)} = -T^2 D'' \nabla x_1 - L\nabla p - \lambda\nabla T. \tag{4.3}$$

The transport coefficients in these equations are the diffusion coefficient D, the pressure diffusion coefficient D_p, the thermal diffusion coefficient D', the shear viscosity η, the Dufour coefficient D'', the pressure energy coefficient L, and the thermal conductivity λ. These coefficients are defined as

$$D = -\frac{\rho}{dm_2 n}\int d\mathbf{v}\,\mathbf{V}\cdot\mathcal{A}_1, \tag{4.4}$$

$$D_p = -\frac{m_1 p}{d\rho}\int d\mathbf{v}\,\mathbf{V}\cdot\mathcal{B}_1, \tag{4.5}$$

$$D' = -\frac{m_1 T}{d\rho}\int d\mathbf{v}\,\mathbf{V}\cdot\mathcal{C}_1, \tag{4.6}$$

$$\eta = -\frac{1}{(d-1)(d+2)}\sum_{i=1}^{2} m_i\int d\mathbf{v}\,\mathbf{V}\mathbf{V}:\mathcal{D}_i, \tag{4.7}$$

$$D'' = -\frac{1}{dT^2}\sum_{i=1}^{2}\frac{m_i}{2}\int d\mathbf{v}\,V^2\mathbf{V}\cdot\mathcal{A}_i, \tag{4.8}$$

$$L = -\frac{1}{d}\sum_{i=1}^{2}\frac{m_i}{2}\int d\mathbf{v}\,V^2\mathbf{V}\cdot\mathcal{B}_i, \tag{4.9}$$

$$\lambda = -\frac{1}{d}\sum_{i=1}^{2}\frac{m_i}{2}\int d\mathbf{v}\,V^2\mathbf{V}\cdot\mathcal{C}_i. \tag{4.10}$$

As for ordinary gases [4], the unknowns $\mathcal{A}_i(\mathbf{V})$, $\mathcal{B}_i(\mathbf{V})$, $\mathcal{C}_i(\mathbf{V})$, and $\mathcal{D}_i(\mathbf{V})$ are the solutions of the following set of coupled linear integral equations:

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_1]\mathcal{A}_1 + \mathcal{M}_1\mathcal{A}_2 = \mathbf{A}_1 + \left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T}(p\mathcal{B}_1 + T\mathcal{C}_1), \tag{4.11a}$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_2]\mathcal{A}_2 + \mathcal{M}_2\mathcal{A}_1 = \mathbf{A}_2 + \left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T}(p\mathcal{B}_2 + T\mathcal{C}_2), \quad (4.11b)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_1 - 2\zeta^{(0)}]\mathcal{B}_1 + \mathcal{M}_1\mathcal{B}_2 = \mathbf{B}_1 + \frac{T\zeta^{(0)}}{p}\mathcal{C}_1, \quad (4.12a)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_2 - 2\zeta^{(0)}]\mathcal{B}_2 + \mathcal{M}_2\mathcal{B}_1 = \mathbf{B}_2 + \frac{T\zeta^{(0)}}{p}\mathcal{C}_2, \quad (4.12b)$$

$$\left[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_1 - \frac{1}{2}\zeta^{(0)}\right]\mathcal{C}_1 + \mathcal{M}_1\mathcal{C}_2 = \mathbf{C}_1 - \frac{p\zeta^{(0)}}{2T}\mathcal{B}_1, \quad (4.13a)$$

$$\left[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_2 - \frac{1}{2}\zeta^{(0)}\right]\mathcal{C}_2 + \mathcal{M}_2\mathcal{C}_1 = \mathbf{C}_2 - \frac{p\zeta^{(0)}}{2T}\mathcal{B}_2, \quad (4.13b)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_1]\mathcal{D}_1 + \mathcal{M}_1\mathcal{D}_2 = \mathsf{D}_1, \quad (4.14a)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_2]\mathcal{D}_2 + \mathcal{M}_2\mathcal{D}_1 = \mathsf{D}_2. \quad (4.14b)$$

In the above equations, the quantities $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, and D_i are given by (7.8–7.11), respectively. They depend on the local HCS distribution $f_i^{(0)}$. In addition, we have introduced the linearized Boltzmann collision operators

$$\mathcal{L}_1 X = -(J_{11}[f_1^{(0)}, X] + J_{11}[X, f_1^{(0)}] + J_{12}[X, f_2^{(0)}]), \quad (4.15)$$

$$\mathcal{M}_1 X = -J_{12}[f_1^{(0)}, X]. \quad (4.16)$$

The corresponding expressions for the operators $\mathcal{L}_2$ and $\mathcal{M}_2$ can be easily obtained from (4.15) and (4.16) by just making the changes $1 \leftrightarrow 2$. Note that in (4.11) the cooling rate $\zeta^{(0)}$ depends on x_1 explicitly and through its dependence on $\gamma(x_1)$. This dependence gives rise to significant new contributions to the integral equations for the transport coefficients. Furthermore, the external field does not occur in (4.11–4.14). This is because the particular form of the gravitational force.

4.1 Sonine Polynomial Approximation

So far, all the results are exact. However, explicit expressions for the NS transport coefficients requires to solve (4.11–4.14) as well as the integral equations (3.9) for the reference distributions $f_i^{(0)}$. As said before, the results obtained in the HCS [12] have shown that $f_i^{(0)}$ is well represented by its Maxwellian form (3.11) in the region of thermal velocities. For this reason and to provide simple but accurate expressions for the transport coefficients, non-Gaussian corrections to $f_i^{(0)}$ will be neglected in our theory. The full expressions for the transport coefficients in the case $d = 3$ (including non-Gaussian corrections) can be found in Ref. [26]. It must remarked that while the effect of these non-Gaussian corrections on the transport coefficients is not important in the case of the mass flux and the pressure tensor, the same does not happen for the heat flux, where the influence of them is not negligible at high inelasticity [28]. With respect to the functions $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i)$, we will expand them in a series expansion of Sonine polynomials and will consider only the leading terms. The procedure is described in Appendix 2 and only the final expressions will be provided here.

4.2 Mass Flux

In dimensionless form, the transport coefficients associated with the mass flux, D, D_p, and D' can be written as

$$D = \frac{\rho T}{m_1 m_2 v_0} D^*, \qquad D_p = \frac{nT}{\rho v_0} D_p^*, \qquad D' = \frac{nT}{\rho v_0} D'^*, \tag{4.17}$$

where $v_0 = n\sigma_{12}^{d-1} v_0$ is an effective collision frequency. The explicit forms are

$$D^* = \left(v^* - \frac{1}{2}\zeta^*\right)^{-1} \left[\left(\frac{\partial}{\partial x_1} x_1 \gamma_1\right)_{p,T} + \left(\frac{\partial \zeta^*}{\partial x_1}\right)_{p,T} \left(1 - \frac{\zeta^*}{2v^*}\right) D_p^*\right], \tag{4.18}$$

$$D_p^* = x_1 \left(\gamma_1 - \frac{\mu}{x_2 + \mu x_1}\right) \left(v^* - \frac{3}{2}\zeta^* + \frac{\zeta^{*2}}{2v^*}\right)^{-1}, \tag{4.19}$$

$$D'^* = -\frac{\zeta^*}{2v^*} D_p^*. \tag{4.20}$$

Here, $\zeta^* = \zeta^{(0)}/v_0$, and v^* is given by

$$v^* = \frac{2\pi^{(d-1)/2}}{d\Gamma(\frac{d}{2})} (1 + \alpha_{12}) \left(\frac{\theta_1 + \theta_2}{\theta_1 \theta_2}\right)^{1/2} (x_2 \mu_{21} + x_1 \mu_{12}). \tag{4.21}$$

Since $\mathbf{j}_1^{(1)} = -\mathbf{j}_2^{(1)}$ and $\nabla x_1 = -\nabla x_2$, D^* must be symmetric while D_p^* and D'^* must be antisymmetric with respect to the exchange $1 \leftrightarrow 2$. This can be easily verified by noting that $x_1 \gamma_1 + x_2 \gamma_2 = 1$. The expressions for D^*, D_p^* and D'^* reduce to those recently obtained [21, 42] in the tracer limit ($x_1 \to 0$).

4.3 Pressure Tensor

The shear viscosity coefficient η can be written as

$$\eta = \frac{p}{v_0} (x_1 \gamma_1^2 \eta_1^* + x_2 \gamma_2^2 \eta_2^*), \tag{4.22}$$

where the expression of the (dimensionless) partial contribution η_i^* is

$$\eta_1^* = 2 \frac{\gamma_2(2\tau_{22} - \zeta^*) - 2\gamma_1 \tau_{12}}{\gamma_1 \gamma_2[\zeta^* - 2\zeta^*(\tau_{11} + \tau_{22}) + 4(\tau_{11}\tau_{22} - \tau_{12}\tau_{21})]}. \tag{4.23}$$

Here, we have introduced the (reduced) collision frequencies τ_{11} and τ_{12} given by

$$\tau_{11} = \frac{2\pi^{(d-1)/2}}{d(d+2)\Gamma(\frac{d}{2})} \left\{x_1 \left(\frac{\sigma_1}{\sigma_{12}}\right)^{d-1} (2\theta_1)^{-1/2}(3 + 2d - 3\alpha_{11})(1 + \alpha_{11})\right.$$

$$+ 2x_2 \mu_{21}(1 + \alpha_{12})\theta_1^{3/2}\theta_2^{-1/2}\left[(d+3)(\mu_{12}\theta_2 - \mu_{21}\theta_1)\theta_1^{-2}(\theta_1 + \theta_2)^{-1/2}\right.$$

$$\left.\left. + \frac{3 + 2d - 3\alpha_{12}}{2}\mu_{21}\theta_1^{-2}(\theta_1 + \theta_2)^{1/2} + \frac{2d(d+1) - 4}{2(d-1)}\theta_1^{-1}(\theta_1 + \theta_2)^{-1/2}\right]\right\}, \tag{4.24}$$

$$\tau_{12} = \frac{4\pi^{(d-1)/2}}{d(d+2)\Gamma(\frac{d}{2})} x_2 \frac{\mu_{21}^2}{\mu_{12}} \theta_1^{3/2} \theta_2^{-1/2} (1+\alpha_{12})$$

$$\times \left[(d+3)(\mu_{12}\theta_2 - \mu_{21}\theta_1)\theta_2^{-2}(\theta_1 + \theta_2)^{-1/2} \right.$$

$$\left. + \frac{3+2d-3\alpha_{12}}{2} \mu_{21}\theta_2^{-2}(\theta_1+\theta_2)^{1/2} - \frac{2d(d+1)-4}{2(d-1)} \theta_2^{-1}(\theta_1+\theta_2)^{-1/2} \right]. \quad (4.25)$$

A similar expression can be obtained for η_2^* by just making the changes $1 \leftrightarrow 2$.

4.4 Heat Flux

The case of the heat flux is more involved since it requires to consider the second Sonine approximation. The transport coefficients appearing in the heat flux, D'', L, and λ can be written as

$$D'' = -\frac{d+2}{2} \frac{n}{(m_1+m_2)v_0} \left[\frac{x_1\gamma_1^3}{\mu_{12}} d_1^* + \frac{x_2\gamma_2^3}{\mu_{21}} d_2^* - \left(\frac{\gamma_1}{\mu_{12}} - \frac{\gamma_2}{\mu_{21}} \right) D^* \right], \quad (4.26)$$

$$L = -\frac{d+2}{2} \frac{T}{(m_1+m_2)v_0} \left[\frac{x_1\gamma_1^3}{\mu_{12}} \ell_1^* + \frac{x_2\gamma_2^3}{\mu_{21}} \ell_2^* - \left(\frac{\gamma_1}{\mu_{12}} - \frac{\gamma_2}{\mu_{21}} \right) D_p^* \right], \quad (4.27)$$

$$\lambda = -\frac{d+2}{2} \frac{nT}{(m_1+m_2)v_0} \left[\frac{x_1\gamma_1^3}{\mu_{12}} \lambda_1^* + \frac{x_2\gamma_2^3}{\mu_{21}} \lambda_2^* - \left(\frac{\gamma_1}{\mu_{12}} - \frac{\gamma_2}{\mu_{21}} \right) D'^* \right], \quad (4.28)$$

where the coefficients D^*, D_p^*, and D'^* are given by (4.18–4.20), respectively. The expressions of the (dimensionless) Sonine coefficients d_i^*, ℓ_i^*, and λ_i^* are

$$d_1^* = \frac{1}{\Delta} \left\{ 2[2\nu_{12}Y_2 - Y_1(2\nu_{22}-3\zeta^*)][\nu_{12}\nu_{21} - \nu_{11}\nu_{22} + 2(\nu_{11}+\nu_{22})\zeta^* - 4\zeta^{*2}] \right.$$

$$+ 2\left(\frac{\partial\zeta^*}{\partial x_1} \right)_{p,T} (Y_3 + Y_5)[2\nu_{12}\nu_{21} + 2\nu_{22}^2 - \zeta^*(7\nu_{22}-6\zeta^*)]$$

$$\left. - 2\nu_{12}\left(\frac{\partial\zeta^*}{\partial x_1} \right)_{p,T} (Y_4 + Y_6)(2\nu_{11}+2\nu_{22}-7\zeta^*) \right\}, \quad (4.29)$$

$$\ell_1^* = \frac{1}{\Delta} \{ -2Y_3[2(\nu_{12}\nu_{21} - \nu_{11}\nu_{22})\nu_{22}$$

$$+ \zeta^*(7\nu_{11}\nu_{22} - 5\nu_{12}\nu_{21} + 2\nu_{22}^2 - 6\nu_{11}\zeta^* - 7\nu_{22}\zeta^* + 6\zeta^{*2})]$$

$$+ 2Y_4\nu_{12}[2\nu_{12}\nu_{21} - 2\nu_{11}\nu_{22} + 2\zeta^*(\nu_{11}+\nu_{22}) - \zeta^{*2}]$$

$$+ 2Y_5\zeta^*[2\nu_{12}\nu_{21} + \nu_{22}(2\nu_{22}-7\zeta^*) + 6\zeta^{*2}]$$

$$- 2\nu_{12}\zeta^*Y_6[2(\nu_{11}+\nu_{22}) - 7\zeta^*] \}, \quad (4.30)$$

$$\lambda_1^* = \frac{1}{\Delta}\{-Y_3\zeta^*[2\nu_{12}\nu_{21} + \nu_{22}(2\nu_{22} - 7\zeta^*) + 6\zeta^{*2}] + \nu_{12}\zeta^*Y_4[2(\nu_{11} + \nu_{22}) - 7\zeta^*]$$

$$-Y_5[4\nu_{12}\nu_{21}(\nu_{22} - \zeta^*) + 2\nu_{22}^2(5\zeta^* - 2\nu_{11})$$

$$+ 2\nu_{11}(7\nu_{22}\zeta^* - 6\zeta^{*2}) + 5\zeta^{*2}(6\zeta^* - 7\nu_{22})]$$

$$+ \nu_{12}Y_6[4\nu_{12}\nu_{21} + 2\nu_{11}(5\zeta^* - 2\nu_{22}) + \zeta^*(10\nu_{22} - 23\zeta^*)]\}, \tag{4.31}$$

where

$$\Delta = [4(\nu_{12}\nu_{21} - \nu_{11}\nu_{22}) + 6\zeta^*(\nu_{11} + \nu_{22}) - 9\zeta^{*2}]$$

$$\times [\nu_{12}\nu_{21} - \nu_{11}\nu_{22} + 2\zeta^*(\nu_{11} + \nu_{22}) - 4\zeta^{*2}]. \tag{4.32}$$

In the above equations, the Y's are defined by (8.27–8.29), while the (reduced) collision frequencies ν_{11} and ν_{12} are given by (9.17) and (9.18), respectively. The expressions for d_2^*, ℓ_2^*, and λ_2^* can be obtained from (4.29–4.32) by setting $1 \leftrightarrow 2$. As expected, our results for the heat flux show that D'' is antisymmetric with respect to the change $1 \leftrightarrow 2$, while L and λ are symmetric. Consequently, in the case of mechanically equivalent particles ($m_1 = m_2 \equiv m$, $\sigma_1 = \sigma_2 \equiv \sigma$, $\alpha_{ij} \equiv \alpha$), the coefficient D'' vanishes.

In the three-dimensional case ($d = 3$), all the above expressions for the transport coefficients reduce to those previously derived for hard spheres [26, 28] when one takes Maxwellian distributions (3.11) for the zeroth-order approximations $f_i^{(0)}$. For mechanically equivalent particles, the results obtained by Brey and Cubero [43] for a d-dimensional mono-component gas are also recovered. This confirms the self-consistency of the results derived here.

4.5 Comparison with Other Theories

Before checking the accuracy of our expressions by comparing them with computer simulations, it is instructive first to make some comparison with previous results [8, 31]. These results assume energy equipartition ($T_i = T$) and so, they are based on a standard CE expansion around the Maxwellian (3.15) instead of the local HCS distribution. Figures 2 and 3 show the dependence of the reduced pressure diffusion coefficient $D_p(\alpha)/D_p(1)$ and the reduced thermal conductivity coefficient $\lambda(\alpha)/\lambda(1)$, respectively, as a function of the (common) coefficient of restitution $\alpha_{ij} \equiv \alpha$ for $d = 3$, $\omega = 1$, $x_1 = 0.2$, and two different mass ratios μ: $\mu = 0.5$ (a) and $\mu = 4$ (b). Here, $D_p(1)$ and $\lambda(1)$ are the values of D_p and λ for

Fig. 2 Plot of the reduced pressure diffusion coefficient $D_p(\alpha)/D_p(1)$ as a function of the (common) coefficient of restitution α for binary mixtures with $x_1 = 0.2$, $\omega = 1$ in the case of a three-dimensional system ($d = 3$) and two values of the mass ratio μ: $\mu = 0.5$ (a) and $\mu = 4$ (b). The solid lines refer to the results derived here and the dashed lines correspond to the results assuming the equality of the partial temperatures
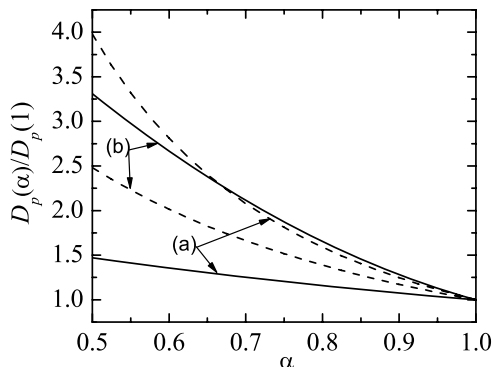
Fig. 3 Plot of the reduced thermal conductivity coefficient $\lambda(\alpha)/\lambda(1)$ as a function of the (common) coefficient of restitution α for binary mixtures with $x_1 = 0.2$, $\omega = 1$ in the case of a three-dimensional system ($d = 3$) and two values of the mass ratio μ: $\mu = 0.5$ (a) and $\mu = 4$ (b). The solid lines refer to the results derived here and the dashed lines correspond to the results assuming the equality of the partial temperatures
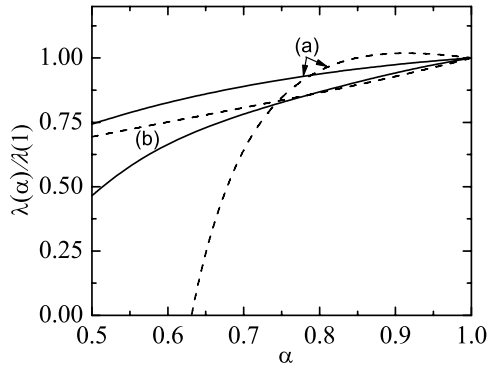
elastic collisions. We see that the deviation from the functional form for elastic collisions is quite important in both theories, even for moderate dissipation. It is apparent that the dependence of the transport coefficients on dissipation is quantitatively different in both models, especially at strong dissipation (say for instance, $\alpha = 0.5$). This clearly shows the real quantitative effect of two different species temperatures on transport in granular mixtures.

5 Comparison with Monte Carlo Simulations

A said before, the expressions derived in Sect. 4 for the NS transport coefficients have been obtained by considering two different approximations. First, since the deviation of $f_i^{(0)}$ from its Maxwellian form (3.11) is quite small in the region of thermal velocities, we have used the Maxwellian distribution (3.11) as a trial function for $f_i^{(0)}$. Second, we have only considered the leading terms of an expansion of the distribution $f_i^{(1)}$ in Sonine polynomials. Both approximations allow one to offer a simplified kinetic theory for a d-dimensional granular binary mixture. To check the accuracy of the above predictions, in this section we numerically solve the Boltzmann equation by means of the DSMC method [38] and compare theory and simulation in the cases of the diffusion coefficient D (in the tracer limit) and the shear viscosity coefficient η. Previous comparisons carried out for hard spheres [29, 30] when one takes into account the deviations of $f_i^{(0)}$ from their Maxwellian forms have shown good agreement between theory and simulation, even for strong dissipation (say $\alpha \gtrsim 0.5$). Here, we expect that such good agreement is also maintained in the case of hard disks when one replaces $f_i^{(0)} \rightarrow f_{i,M}$. Let us study each coefficient separately.

5.1 Tracer Diffusion Coefficient

We consider the special case in which one of the components of the mixture (say, for instance, species 1) is present in tracer concentration ($x_1 \rightarrow 0$). In this situation, $\partial \zeta^*/\partial x_1 \rightarrow 0$ and so, the expression (4.18) for the reduced diffusion coefficient D^* becomes

$$D^* = \frac{\gamma}{\nu^* - \frac{1}{2}\zeta^*}, \tag{5.1}$$

where now

$$\zeta^* = \frac{\pi^{(d-1)/2}}{d\Gamma(d/2)}\left(\frac{\sigma_2}{\sigma_{12}}\right)^{d-1}\sqrt{2\mu_{12}}(1 - \alpha_{22}^2), \tag{5.2}$$

$$\nu^* = \frac{2\pi^{(d-1)/2}}{d\Gamma(d/2)}\mu_{21}\sqrt{\mu_{12}+\mu_{21}}\gamma(1+\alpha_{12}). \tag{5.3}$$

The diffusion coefficient of impurities in a granular gas undergoing homogeneous cooling state can be measured in simulation from the mean square displacement of the tracer particle after a time interval t [29, 44]:

$$\frac{\partial}{\partial t}\langle|\mathbf{r}(t)-\mathbf{r}(0)|^2\rangle = \frac{2dD}{n_2}. \tag{5.4}$$

Equation (5.4) is the Einstein form. This relation (written in appropriate dimensionless variables to eliminate the time dependence of $D(t)$) was used in Ref. [29] to measure the diffusion coefficient for hard spheres. More details on this procedure can be found in Ref. [29].

If the hydrodynamic description (or normal solution in the context of the CE method) applies, then the diffusion coefficient $D(t)$ depends on time only through its dependence on the temperature $T(t)$. In this case, after a transient regime, the reduced diffusion coefficient $D^* = (m_1 m_2/\rho)D(t)v_0(t)/T(t)$ achieves a time-independent value. Here, we compare the steady state values of D^* obtained from Monte Carlo simulations with the theoretical predictions given by the first Sonine approximation (5.1). The dependence of D^* on the common coefficient of restitution $\alpha_{ij} \equiv \alpha$ is shown in Fig. 4 in the case of hard disks for three different systems. The symbols refer to computer simulations while the lines correspond to the kinetic theory results given by (5.1). Molecular dynamics (MD) results reported in Ref. [45] when impurities and particles of the gas are mechanically equivalent have also been included. We observe that MD and DSMC results for $\mu = \omega = 1$ are consistent among themselves in the range of values of α explored. This good agreement gives support to the applicability of the inelastic Boltzmann equation beyond the quasielastic limit. It is apparent that the agreement between the first Sonine approximation and simulation results is excellent when impurities and particles of the gas are mechanically equivalent and when impurities are much heavier and/or much larger than the particles of the gas (Brownian limit). However, some discrepancies between simulation an theory are found with decreasing values of the mass ratio m_1/m_2 and the size ratio σ_1/σ_2. These discrepancies are not easily observed in Fig. 4 because of the small magnitude of D^* for $\mu = 1/4$. The above findings agree with those previously reported for hard spheres [29], where it was shown that the second Sonine approximation improves the qualitative predictions from the first Sonine approximation for the cases in which the gas

Fig. 4 Plot of the reduced diffusion coefficient D^* as a function of the (common) coefficient of restitution α for binary mixtures with $\omega = \mu$ in the case of a two-dimensional system ($d = 2$). The symbols are computer simulation results obtained from the mean square displacement and the lines are the theoretical results obtained in the first Sonine approximation. The DSMC results correspond to $\mu = 1/4$ (●), $\mu = 4$ (○) and $\mu = 1$ (◇). Molecular dynamics results reported in Ref. [45] for $\mu = 1$ (△) have also been included
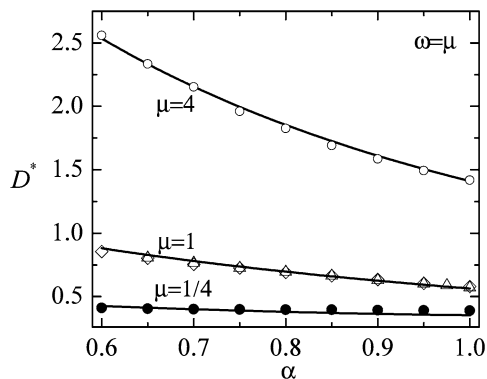
Fig. 5 Plot of the reduced shear
viscosity $\eta^*(\alpha) = \eta(\alpha)/\eta(1)$ as a
function of the (common)
coefficient of restitution α for
binary mixtures constituted by
particles of the same mass
density ($\omega = \mu^{1/2}$) in the case of
a two-dimensional system
($d = 2$). The symbols are
computer simulation results and
the lines are the theoretical
results obtained in the first
Sonine approximation. The
DSMC results correspond to
$\mu = 1$ (O), $\mu = 4$ (●) and $\mu = 8$
(◇). We have also included
DSMC results obtained in
Ref. [48] for $\mu = 1$ (△) from the
Green–Kubo relation
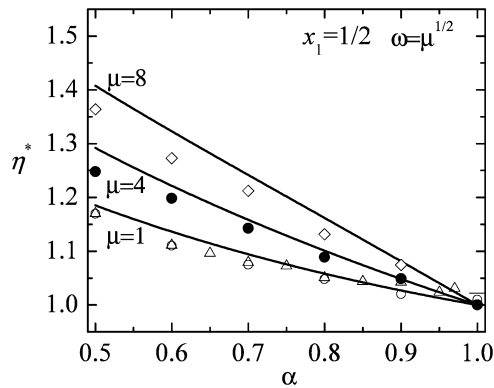
particles are heavier and/or larger than impurities. The comparison carried out here for disks
confirms the above expectations and shows that the Sonine polynomial expansion exhibits a
slow convergence for sufficiently small values of the mass ratio μ and/or the size ratio ω.

5.2 Shear Viscosity Coefficient

The shear viscosity η is perhaps the most widely studied transport coefficient in granular flu-
ids. In the case of granular mixtures, this coefficient has been measured [30] when the system
is *heated* by the action of an external driving force (thermostat) that exactly compensates for
cooling effects associated with dissipation of collisions. The corresponding shear viscosity
of the mixture (which slightly differs from the one obtained in the free cooling case) has
been determined by means of the CE method in the low-density regime [30] as well as for a
moderate dense mixture [46]. The theoretical predictions compare reasonably well with the
corresponding numerical solutions of the Boltzmann and Enskog kinetic equations.

More recently, a new alternative method has been proposed to measure the (true) NS
shear viscosity coefficient [47]. The method is based on the simple shear flow state modified
by the introduction of a deterministic non-conservative force (which compensates for the
collisional cooling) along with a stochastic process. While the external force is introduced to
allow the granular fluid to approach a Newtonian regime, the stochastic process is introduced
to mimic the conditions appearing in the CE method to NS order. Although the method
was originally devised to a single granular gas, its extension to multicomponent systems is
straightforward. Here, we use this procedure to measure the shear viscosity of the mixture
by means of the DSMC method. More technical details on this procedure and its application
to dense gases can be found in Ref. [47].

Comparison between the first Sonine approximation and computer simulations for
$\eta^*(\alpha) = \eta(\alpha)/\eta(1)$ is shown in Fig. 5 for three different mixtures constituted by particles
of the same mass density (i.e., $\mu = \omega^d$) in the case of a two-dimensional system. Here, $\eta(1)$
refers to the elastic value for the shear viscosity coefficient and we have assumed again a
common value of the coefficient of restitution α. The symbols represent the simulation data
and the lines correspond to the theoretical results. We have also included recent simulation
results [48] for η^* obtained from the Green–Kubo relation in the one-component case. Good
agreement among the data presented here and those reported in Ref. [48] for $\mu = 1$ is ob-
served. In addition, as happens for hard spheres [30], we see that in general the agreement

between the first Sonine approximation and simulation is quite good. At a quantitative level, the theory slightly overestimates the simulation data, especially for strong dissipation and for mixtures of particles of different masses and/or sizes. However, such discrepancies are quite small since for instance, they are smaller than 3% at $\alpha = 0.5$ for $\mu = 8$ and $\omega = \sqrt{8}$. This shows again the reliability of the first Sonine approximation for the shear viscosity coefficient. It must be noted that this conclusion cannot in principle be extended to the transport coefficients associated with the heat flux since recent comparisons for a single gas [48–50] have shown significant discrepancies between the first Sonine approximation and computer simulations for high inelasticity (say $\alpha \lesssim 0.7$). In this case, the agreement between theory and simulation can be significantly improved by the use of a modified first Sonine approximation [51].

6 Discussion

The main objective of this work has been to obtain the NS transport coefficients of a granular binary mixture at low density. In contrast to previous works [8, 31], the present study is based on a modified CE solution of the inelastic Boltzmann equation that takes into account non-equipartition of energy. There is no phenomenology involved as the equations and the transport coefficients have been derived systematically from the inelastic Boltzmann equation by the CE expansion around the local HCS. Since the spatial gradients are assumed to be independent of the coefficients of restitution, although the NS equations restrict their applicability to first order in gradients the corresponding transport coefficients hold *a priori* to arbitrary degree of inelasticity. All the calculations have been performed in an arbitrary number d of dimensions, previous results [26] being recovered for $d = 3$.

The constitutive equations to NS order for the mass flux, the stress tensor, and the heat flux are given by (4.1–4.3), respectively. The associated transport coefficients are the mutual diffusion coefficient D, the pressure diffusion coefficient D_p, and the thermal diffusion coefficient D' in the case of the mass flux, the shear viscosity coefficient η for the pressure tensor, and the Dufour coefficient D'', the pressure energy coefficient L, and the thermal conductivity λ in the case of the heat flux. These coefficients are determined from the solutions of the set of coupled linear integral equations (4.11–4.14). In addition, the NS transport coefficients also depend on the reference distributions $f_i^{(0)}$, which are not Maxwellians because they obey the integral equations (3.9). To solve the above integral equations and provide good estimates for the transport coefficients, we have considered two approximations: (i) the distributions $f_i^{(0)}$ have been replaced by their Maxwellian forms (3.11) at the temperature T_i for that species and, (ii) we have only considered the leading terms in a series of Sonine polynomials for the first-order distribution $f_i^{(1)}$. By using both approximations, explicit expressions of the *seven* NS transport coefficients have been obtained as functions of the coefficients of restitution and the concentration and the ratios of mass and diameters. In dimensionless forms, the coefficients D, D_p, and D' are given by (4.18–4.20), respectively, the shear viscosity η is given by (4.22) and (4.23), while the expressions of the coefficients D'', L, and λ are provided by (4.26–4.32).

Previous results [8–11] derived from the CE method have typically introduced additional assumptions for convenience that are not internally consistent with constructing a solution to the Boltzmann equation. Thus, in most of the cases the reference state $f_i^{(0)}$ has been chosen to be a Maxwellian at the same temperature [see (3.15)]. This assumption is presumed to give accurate results at weak dissipation where energy equipartition can be still considered as a good approximation. However, as shown in Fig. 1, the temperature ratio T_1/T_2 clearly differs

Fig. 6 Plot of the reduced shear viscosity coefficient $\eta^* = \eta(\alpha)/\eta(1)$ as a function of the (common) coefficient of restitution α for binary mixtures with $x_1 = 0.2$, $\omega = 1$ and two values of the mass ratio μ: $\mu = 0.5$ (a) and $\mu = 4$ (b). The solid lines refer to spheres ($d = 3$) while the dashed lines correspond to disks ($d = 2$)
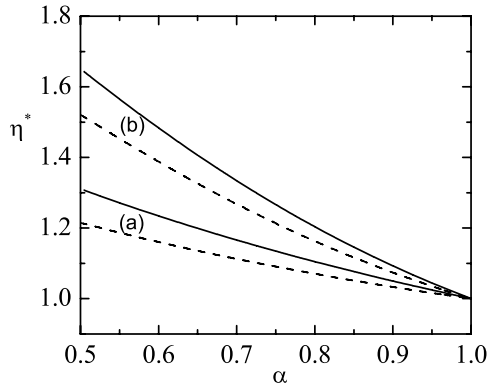
from 1 as dissipation increases. Here, we have replaced $f_i^{(0)} \to f_{i,M}$ so that, the influence of the fourth-cumulants c_i of $f_i^{(0)}$ has been ignored [12]. Comparison between the expressions derived in this paper by taking the approximation (3.11) with those obtained by assuming energy equipartition shows important discrepancies as the coefficient of restitution decreases [see Figs. 2 and 3]. Moreover, as an added value of our theory, the use of the Maxwellian approximation (3.11) for $f_i^{(0)}$ allows one to provide simple and explicit expressions for all the transport coefficients in terms of the parameters of the mixture. This contrasts with the relatively recent study for hard spheres [26] where the constitutive relations for the fluxes were not explicitly displayed.

As a complementary route and to check the reliability of our theory, the analytical results derived for the diffusion coefficient D and the shear viscosity η in the first Sonine approximation have been compared with those obtained from numerical solutions of the Boltzmann equation by means of the DSMC method for a two-dimensional system. For the sake of simplicity, all the simulations have considered a common coefficient of restitution $\alpha \equiv \alpha_{ij}$. As expected, theory and simulation clearly show that the influence of dissipation on mass and momentum transport is quite important since there is a relevant dependence of the diffusion D and viscosity η coefficients on α. With respect to the accuracy of the theoretical predictions, we see that in general the CE results in the first Sonine approximation exhibit a good agreement with the simulation data. Exceptions to this agreement are extreme mass or size ratios and strong dissipation. These discrepancies are basically due to the use of the first Sonine approximation and can be partially mitigated by considering the second and third Sonine approximations [29] or the use of a modified first Sonine approximation [51].

As said in the Introduction, the results obtained in this paper are of great practical interest since most of the experiments and simulations are performed in two dimensions. On the other hand, apart from this practical interest, the knowledge of the NS transport coefficients of a d-dimensional mixture allows one to investigate the influence of dimensionality on the transport properties of the system. To illustrate this effect, in Fig. 6 we plot the reduced shear viscosity $\eta^* \equiv \eta(\alpha)/\eta(1)$ versus the coefficient of restitution α for $\omega = 1$, $x_1 = 0.2$, and two different mass ratios μ: $\mu = 0.5$ (a) and $\mu = 4$ (b). We have considered the physical cases of hard spheres (solid lines) and hard disks (dashed lines). Although the qualitative dependence of η^* on α is quite similar in both systems, we observe that the influence of dissipation on momentum transport is stronger for $d = 3$ than for $d = 2$. This trend is also observed in general in the remaining transport coefficients.

One of the main limitations of our theory is its restriction to dilute gases. In this situation, the collisional transfer contributions to the fluxes are neglected and only their kinetic contri-

butions are considered. Possible extension of the present kinetic theory to higher densities can be done in the context of the revised Enskog theory. Preliminary results [46] have been focused on the uniform shear flow state to get directly the shear viscosity coefficient. The extension of this study [46] to states with gradients of concentration, pressure, and temperature is somewhat intricate due to subtleties associated with the spatial dependence of the pair correlations functions considered in the revised Enskog theory. On the other hand, it must be remarked that many of the collision integrals appearing in the Enskog description are the same as those appearing in the Boltzmann limit so that one can take advantage of the results reported in this paper. We plan to extend the results derived for moderately dense mixtures of smooth *elastic* hard spheres [52] to inelastic collisions in the near future.

Appendix 1: Chapman–Enskog Method

The velocity distribution function $f_1^{(1)}$ obeys the equation

$$(\partial_t^{(0)} + \mathcal{L}_1) f_1^{(1)} + \mathcal{M}_1 f_2^{(1)} = -\left(\partial_t^{(1)} + \mathbf{v} \cdot \nabla + \mathbf{g} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f_1^{(0)}, \tag{7.1}$$

where the linear operators $\mathcal{L}_1$ and $\mathcal{M}_1$ are defined by (4.15) and (4.16), respectively. A similar equation can be obtained for $f_2^{(1)}$ by interchanging $1 \leftrightarrow 2$. The action of the time derivatives $\partial_t^{(1)}$ on the hydrodynamic fields is

$$D_t^{(1)} x_1 = 0, \tag{7.2}$$

$$D_t^{(1)} p = -\frac{d+2}{d} p \nabla \cdot \mathbf{u}, \tag{7.3}$$

$$D_t^{(1)} T = -\frac{2T}{d} \nabla \cdot \mathbf{u}, \tag{7.4}$$

$$D_t^{(1)} \mathbf{u} = -\rho^{-1} \nabla p + \mathbf{g}, \tag{7.5}$$

where $D_t^{(1)} = \partial_t^{(1)} + \mathbf{u} \cdot \nabla$ and use has been made of the results $\mathbf{j}_i^{(0)} = \mathbf{q}^{(0)} = \zeta^{(1)} = 0$. The last equality follows from the fact that the cooling rate is a scalar, and corrections to first order in the gradients can arise only from the divergence of a vector field. However, as is demonstrated below, there is no contribution to the distribution function proportional to this divergence. We note that this is special to the low density Boltzmann equation and such terms do occur at higher densities [53]. Use of (7.2–7.5) yields

$$-\left(\partial_t^{(1)} + \mathbf{v} \cdot \nabla + \mathbf{g} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f_1^{(0)}$$

$$= -\left(\frac{\partial}{\partial x_1} f_1^{(0)}\right)_{p,T} \mathbf{V} \cdot \nabla x_1 - \left[f_1^{(0)} \mathbf{V} + \frac{nT}{\rho}\left(\frac{\partial}{\partial \mathbf{V}} f_1^{(0)}\right)\right] \cdot \nabla \ln p$$

$$+ \left[f_1^{(0)} + \frac{1}{2} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_1^{(0)}) \right] \mathbf{V} \cdot \nabla \ln T$$

$$+ \left(V_k \frac{\partial}{\partial V_\ell} f_1^{(0)} - \frac{1}{d} \delta_{k\ell} \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{V}} f_1^{(0)} \right) \nabla_k u_\ell. \tag{7.6}$$

Note that the external field does not appear in the right-hand side of (7.6). This is due to the particular form of the gravitational force. Using (7.6), (7.1) can be written as

$$(\partial_t^{(0)} + \mathcal{L}_1) f_1^{(1)} + \mathcal{M}_1 f_2^{(1)} = \mathbf{A}_1 \cdot \nabla x_1 + \mathbf{B}_1 \cdot \nabla p + \mathbf{C}_1 \cdot \nabla T + \mathsf{D}_1 : \nabla \mathbf{u}, \tag{7.7}$$

where

$$\mathbf{A}_i(\mathbf{V}) = - \left(\frac{\partial}{\partial x_1} f_i^{(0)} \right)_{p,T} \mathbf{V}, \tag{7.8}$$

$$\mathbf{B}_i(\mathbf{V}) = -\frac{1}{p} \left[f_i^{(0)} \mathbf{V} + \frac{nT}{\rho} \left(\frac{\partial}{\partial \mathbf{V}} f_i^{(0)} \right) \right], \tag{7.9}$$

$$\mathbf{C}_i(\mathbf{V}) = \frac{1}{T} \left[f_i^{(0)} + \frac{1}{2} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f_i^{(0)}) \right] \mathbf{V}, \tag{7.10}$$

$$\mathsf{D}_i(\mathbf{V}) = \mathbf{V} \frac{\partial}{\partial \mathbf{V}} f_i^{(0)} - \frac{1}{d} \mathbb{1} \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{V}} f_i^{(0)}. \tag{7.11}$$

In (7.8–7.11) it is understood that $i = 1, 2$ and $\mathbb{1}$ is the unit tensor in d dimensions. Note that the trace of D_i vanishes, confirming that the distribution function does not have contribution from the divergence of the flow field. The solutions to (7.7) are of the form

$$f_i^{(1)} = \boldsymbol{\mathcal{A}}_i \cdot \nabla x_1 + \boldsymbol{\mathcal{B}}_i \cdot \nabla p + \boldsymbol{\mathcal{C}}_i \cdot \nabla T + \boldsymbol{\mathcal{D}}_{i,k\ell} \nabla_k u_\ell. \tag{7.12}$$

The coefficients $\boldsymbol{\mathcal{A}}_i$, $\boldsymbol{\mathcal{B}}_i$, $\boldsymbol{\mathcal{C}}_i$, and $\boldsymbol{\mathcal{D}}_i$ are functions of the peculiar velocity $\mathbf{V}$ and the hydrodynamic fields. The cooling rate depends on space through its dependence on x_1, p, and T. The time derivative $\partial_t^{(0)}$ acting on these quantities can be evaluated by the replacement $\partial_t^{(0)} \to -\zeta^{(0)} (T \partial_T + p \partial_p)$. In addition, there are contributions from $\partial_t^{(0)}$ acting on the temperature and pressure gradients given by

$$\partial_t^{(0)} \nabla T = -\nabla(T \zeta^{(0)}) = -\zeta^{(0)} \nabla T - T \nabla \zeta^{(0)}$$

$$= -\frac{\zeta^{(0)}}{2} \nabla T - T \left[\left(\frac{\partial \zeta^{(0)}}{\partial x_1} \right)_{p,T} \nabla x_1 + \frac{\zeta^{(0)}}{p} \nabla p \right], \tag{7.13}$$

$$\partial_t^{(0)} \nabla p = -\nabla(p \zeta^{(0)}) = -\zeta^{(0)} \nabla p - p \nabla \zeta^{(0)}$$

$$= -2 \zeta^{(0)} \nabla p - p \left[\left(\frac{\partial \zeta^{(0)}}{\partial x_1} \right)_{p,T} \nabla x_1 - \frac{\zeta^{(0)}}{2T} \nabla T \right]. \tag{7.14}$$

The corresponding integral equations for the unknowns $\boldsymbol{\mathcal{A}}_i$, $\boldsymbol{\mathcal{B}}_i$, $\boldsymbol{\mathcal{C}}_i$, and $\boldsymbol{\mathcal{D}}_i$ are identified as the coefficients of the independent gradients in (7.12). This leads to (4.11–4.14).

Appendix 2: Leading Sonine Approximations

In this Appendix, we get the explicit expressions of the mass, momentum, and heat fluxes in the first Sonine approximation and neglecting the non-Gaussian corrections to the reference distributions $f_i^{(0)}$ (i.e., the cumulants $c_i = 0$). The procedure to get the leading order contributions in the Sonine polynomial expansion to the transport coefficients is quite similar to the one previously used in the three-dimensional case [26]. Only some partial results will be presented here.

Leading Sonine Approximation to Mass Flux

In the case of the mass flux, the leading Sonine approximations (lowest degree polynomial) of the quantities $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ are

$$\mathcal{A}_1(\mathbf{V}) \to -f_{1,M}\mathbf{V}\frac{m_1 m_2 n}{\rho n_1 T_1}D, \qquad \mathcal{A}_2(\mathbf{V}) \to f_{2,M}\mathbf{V}\frac{m_1 m_2 n}{\rho n_2 T_2}D, \tag{8.1}$$

$$\mathcal{B}_1(\mathbf{V}) \to -f_{1,M}\mathbf{V}\frac{\rho}{p n_1 T_1}D_p, \qquad \mathcal{B}_2(\mathbf{V}) \to f_{2,M}\mathbf{V}\frac{\rho}{p n_2 T_2}D_p, \tag{8.2}$$

$$\mathcal{C}_1(\mathbf{V}) \to -f_{1,M}\mathbf{V}\frac{\rho}{T n_1 T_1}D', \qquad \mathcal{C}_2(\mathbf{V}) \to f_{2,M}\mathbf{V}\frac{\rho}{T n_2 T_2}D', \tag{8.3}$$

where $f_{i,M}$ are the Maxwellian distributions (3.11). Multiplication of (4.11–4.13) by $m_1\mathbf{V}$ and integrating over the velocity yields

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \nu]\left(-\frac{m_1 m_2 n}{\rho}D\right)$$
$$= -\left(\frac{\partial}{\partial x_1}n_1 T_1\right)_{p,T} - \rho\left(\frac{\partial \zeta^{(0)}}{\partial x_1}\right)_{p,T}(D_p + D'), \tag{8.4}$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) - 2\zeta^{(0)} + \nu]\left(-\frac{\rho}{p}D_p\right) = -\frac{n_1 T_1}{p}\left(1 - \frac{m_1 nT}{\rho T_1}\right) - \frac{\rho\zeta^{(0)}}{p}D', \tag{8.5}$$

$$\left[-\zeta^{(0)}(T\partial_T + p\partial_p) - \frac{1}{2}\zeta^{(0)} + \nu\right]\left(-\frac{\rho}{T}D'\right) = \frac{\rho\zeta^{(0)}}{2T}D_p. \tag{8.6}$$

Here, ν is the collision frequency defined by

$$\nu = \frac{1}{d n_1 T_1}\int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot [\mathcal{L}_1(f_{1,M}\mathbf{V}_1) - \delta\gamma\mathcal{M}_1(f_{2,M}\mathbf{V}_2)]$$
$$= -\frac{1}{d n_1 T_1}\int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot (J_{12}[\mathbf{v}_1|f_{1,M}\mathbf{V}_1, f_2^{(0)}] - \delta\gamma J_{12}[\mathbf{v}_1|f_1^{(0)}, f_{2,M}\mathbf{V}_2]), \tag{8.7}$$

where $\delta \equiv x_1/x_2$. The evaluation of the collision integral (8.7) is made in Appendix 3. The self-collision terms of $\mathcal{L}_i$ arising from J_{11} do not occur in (8.7) since they conserve momentum for species 1. From dimensional analysis, $D \sim T^{1/2}$, $D_p \sim pT^{-1/2}$, and $D' \sim pT^{-1/2}$ so the temperature and pressure derivatives can be performed in (8.4–8.6). After performing them, one gets the expressions (4.18), (4.19), and (4.20) for the (reduced) coefficients D^*, D_p^*, and D'^*, respectively.

Leading Sonine Approximation to Pressure Tensor

In the case of the pressure tensor, the leading Sonine approximation for the function $\mathcal{D}_{i,k\ell}$ is

$$\mathcal{D}_{i,k\ell}(\mathbf{V}) \to -f_{i,M}(\mathbf{V}) \frac{\eta_i}{T} R_{i,k\ell}(\mathbf{V}), \quad i = 1, 2 \tag{8.8}$$

where

$$R_{i,k\ell}(\mathbf{V}) = m_i \left(V_k V_\ell - \frac{1}{d} V^2 \delta_{k\ell} \right), \tag{8.9}$$

and

$$\eta_i = -\frac{1}{(d-1)(d+2)} \frac{T}{n_i T_i^2} \int d\mathbf{v} R_{i,k\ell}(\mathbf{V}) \mathcal{D}_{i,k\ell}(\mathbf{V}). \tag{8.10}$$

The shear viscosity η in this approximation can be written as

$$\eta = \frac{p}{v_0} (x_1 \gamma_1^2 \eta_1^* + x_2 \gamma_2^2 \eta_2^*), \tag{8.11}$$

where $\eta_i^* = v_0 \eta_i$. The integral equations for the (reduced) coefficients η_i^* are decoupled from the remaining transport coefficients. The two coefficients η_i^* are obtained by multiplying (4.14) with $R_{i,k\ell}$ and integrating over the velocity to get the coupled set of equations

$$\begin{pmatrix} \tau_{11} - \frac{1}{2}\zeta^* & \tau_{12} \\ \tau_{21} & \tau_{22} - \frac{1}{2}\zeta^* \end{pmatrix} \begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} = \begin{pmatrix} \gamma_1^{-1} \\ \gamma_2^{-1} \end{pmatrix}. \tag{8.12}$$

The (reduced) collision frequencies τ_{ij} are given in terms of the linear collision operator by

$$\tau_{ii} = \frac{1}{(d-1)(d+2)} \frac{1}{n_i T_i^2 v_0} \int d\mathbf{v}_1 R_{i,k\ell} \mathcal{L}_i(f_{i,M} R_{i,k\ell}), \tag{8.13}$$

$$\tau_{ij} = \frac{1}{(d-1)(d+2)} \frac{1}{n_i T_i^2 v_0} \int d\mathbf{v}_1 R_{i,k\ell} \mathcal{M}_i(f_{j,M} R_{j,k\ell}), \quad i \neq j. \tag{8.14}$$

The evaluation of these collision integrals is also given in Appendix 3. The solution of (8.12) is elementary and yields (4.23).

Leading Sonine Approximation to Heat Flux

The heat flux requires going up to the second Sonine approximation. In this case, the quantities $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ are taken to be

$$\begin{aligned} \mathcal{A}_1(\mathbf{V}) &\to f_{1,M} \left[-\frac{m_1 m_2 n}{\rho n_1 T_1} D\mathbf{V} + d_1'' \mathbf{S}_1(\mathbf{V}) \right], \\ \mathcal{A}_2(\mathbf{V}) &\to f_{2,M} \left[\frac{m_1 m_2 n}{\rho n_2 T_2} D\mathbf{V} + d_2'' \mathbf{S}_2(\mathbf{V}) \right], \end{aligned} \tag{8.15}$$

$$\begin{aligned} \mathcal{B}_1(\mathbf{V}) &\to f_{1,M} \left[-\frac{\rho}{p n_1 T_1} D_p \mathbf{V} + \ell_1 \mathbf{S}_1(\mathbf{V}) \right], \\ \mathcal{B}_2(\mathbf{V}) &\to f_{2,M} \left[\frac{\rho}{p n_2 T_2} D_p \mathbf{V} + \ell_2 \mathbf{S}_2(\mathbf{V}) \right], \end{aligned} \tag{8.16}$$

$$\mathcal{C}_1(\mathbf{V}) \to f_{1,M}\left[-\frac{\rho}{Tn_1T_1}D'\mathbf{V} + \lambda_1\mathbf{S}_1(\mathbf{V})\right],$$

$$\mathcal{C}_2(\mathbf{V}) \to f_{2,M}\left[\frac{\rho}{Tn_2T_2}D'\mathbf{V} + \lambda_2\mathbf{S}_2(\mathbf{V})\right], \tag{8.17}$$

where

$$\mathbf{S}_i(\mathbf{V}) = \left(\frac{1}{2}m_iV^2 - \frac{d+2}{2}T_i\right)\mathbf{V}. \tag{8.18}$$

In these equations, it is understood that D, D_p and D' are given by (4.18), (4.19), and (4.20), respectively. The coefficients d_i'', ℓ_i and λ_i are defined as

$$\begin{pmatrix} d_i'' \\ \ell_i \\ \lambda_i \end{pmatrix} = \frac{2}{d(d+2)}\frac{m_i}{n_iT_i^3}\int d\mathbf{v}\mathbf{S}_i(\mathbf{V}) \cdot \begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \end{pmatrix}. \tag{8.19}$$

These coefficients can be determined by multiplying (4.11–4.13) (and their counterparts for the species 2) by $\mathbf{S}_i(\mathbf{V})$ and integrating over the velocity. The final expressions can be obtained by taking into account that $d_1'' \sim T^{-3/2}$, $\ell_1 \sim T^{-3/2}/p$, and $\lambda_1 \sim T^{-5/2}$ and the results

$$\int d\mathbf{v}m_1\mathbf{S}_1(\mathbf{V}) \cdot \mathbf{A}_1 = -\frac{d(d+2)}{4}\frac{n_1T^2}{m_1}\left(\frac{\partial}{\partial x_1}\gamma_1^2\right)_{p,T}, \tag{8.20}$$

$$\int d\mathbf{v}m_1\mathbf{S}_1(\mathbf{V}) \cdot \mathbf{B}_1 = 0, \tag{8.21}$$

$$\int d\mathbf{v}m_1\mathbf{S}_1(\mathbf{V}) \cdot \mathbf{C}_1 = -\frac{d(d+2)}{2}\frac{n_1T_1^2}{m_1T}. \tag{8.22}$$

By using matrix notation, the coupled set of six equations for the quantities

$$\{d_1^*, d_2^*, \ell_1^*, \ell_2^*, \lambda_1^*, \lambda_2^*\} \tag{8.23}$$

can be written as

$$\Lambda_{\sigma\sigma'}X_{\sigma'} = Y_\sigma. \tag{8.24}$$

Here, $d_i^* \equiv T\nu_0 d_i''$, $\ell_i^* \equiv pT\nu_0\ell_i$, $\lambda_i^* \equiv T^2\nu_0\lambda_i$, $X_{\sigma'}$ is the column matrix defined by the set (8.23) and $\Lambda_{\sigma\sigma'}$ is the square matrix

$$\Lambda = \begin{pmatrix} \nu_{11} - \frac{3}{2}\zeta^* & \nu_{12} & -(\frac{\partial\zeta^*}{\partial x_1})_{p,T} & 0 & -(\frac{\partial\zeta^*}{\partial x_1})_{p,T} & 0 \\ \nu_{21} & \nu_{22} - \frac{3}{2}\zeta^* & 0 & -(\frac{\partial\zeta^*}{\partial x_1})_{p,T} & 0 & -(\frac{\partial\zeta^*}{\partial x_1})_{p,T} \\ 0 & 0 & \nu_{11} - \frac{5}{2}\zeta^* & \nu_{12} & -\zeta^* & 0 \\ 0 & 0 & \nu_{21} & \nu_{22} - \frac{5}{2}\zeta^* & 0 & -\zeta^* \\ 0 & 0 & \zeta^*/2 & 0 & \nu_{11} - \zeta^* & \nu_{12} \\ 0 & 0 & 0 & \zeta^*/2 & \nu_{21} & \nu_{22} - \zeta^* \end{pmatrix}. \tag{8.25}$$

The column matrix **Y** is

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}, \tag{8.26}$$

where[1]

$$Y_1 = \frac{D^*}{x_1 \gamma_1^2}(\omega_{12} - \zeta^*) - \frac{1}{\gamma_1^2}\left(\frac{\partial \gamma_1}{\partial x_1}\right)_{p,T},$$

$$Y_2 = -\frac{D^*}{x_2 \gamma_2^2}(\omega_{21} - \zeta^*) - \frac{1}{\gamma_2^2}\left(\frac{\partial \gamma_2}{\partial x_1}\right)_{p,T}, \tag{8.27}$$

$$Y_3 = \frac{D_p^*}{x_1 \gamma_1^2}(\omega_{12} - \zeta^*), \qquad Y_4 = -\frac{D_p^*}{x_2 \gamma_2^2}(\omega_{21} - \zeta^*), \tag{8.28}$$

$$Y_5 = -\frac{1}{\gamma_1} + \frac{D'^*}{x_1 \gamma_1^2}(\omega_{12} - \zeta^*), \qquad Y_6 = -\frac{1}{\gamma_2} - \frac{D'^*}{x_2 \gamma_2^2}(\omega_{21} - \zeta^*). \tag{8.29}$$

Here, we have introduced the (reduced) collision frequencies

$$\nu_{ii} = \frac{2}{d(d+2)} \frac{m_i}{n_i T_i^3 v_0} \int d\mathbf{v}_1 \mathbf{S}_i \cdot \mathcal{L}_i(f_{i,M}\mathbf{S}_i), \tag{8.30}$$

$$\nu_{ij} = \frac{2}{d(d+2)} \frac{m_i}{n_i T_i^3 v_0} \int d\mathbf{v}_1 \mathbf{S}_i \cdot \mathcal{M}_i(f_{j,M}\mathbf{S}_j), \quad i \neq j, \tag{8.31}$$

$$\omega_{12} = \frac{2}{d(d+2)} \frac{m_1}{n_1 T_1^2 v_0}\left[\int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{L}_1(f_{1,M}\mathbf{V}_1) - \delta\gamma \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{M}_1(f_{2,M}\mathbf{V}_2)\right], \tag{8.32}$$

$$\omega_{21} = \frac{2}{d(d+2)} \frac{m_2}{n_2 T_2^2 v_0}\left[\int d\mathbf{v}_1 \mathbf{S}_2 \cdot \mathcal{L}_2(f_{2,M}\mathbf{V}_1) - \frac{1}{\delta\gamma} \int d\mathbf{v}_1 \mathbf{S}_2 \cdot \mathcal{M}_2(f_{1,M}\mathbf{V}_2)\right]. \tag{8.33}$$

The expressions of the collision integrals (8.30), (8.31), and (8.32) are given in Appendix 3. The solution to (8.24) is

$$X_\sigma = (\Lambda^{-1})_{\sigma\sigma'} Y_{\sigma'}. \tag{8.34}$$

From this relation one gets the expressions (4.29), (4.30), and (4.31) for the coefficients d_i^*, ℓ_i^* and λ_i^*, respectively.

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[1]Some misprints occur in the expressions given in Ref. [26] for the heat flux. The results displayed here correct and extend such results to d dimensions

Appendix 3: Collision Integrals

In this appendix we compute the different collision integrals appearing in the expressions of the transport coefficients. To simplify all the integrals, we use the property

$$\int d\mathbf{v}_1 h(\mathbf{V}_1) J_{ij}[\mathbf{v}_1|f_i, f_j] = \sigma_{ij}^{d-1} \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_i(\mathbf{V}_1) f_j(\mathbf{V}_2)$$
$$\times \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})[h(\mathbf{V}_1'') - h(\mathbf{V}_1)], \quad (9.1)$$

with

$$\mathbf{V}_1'' = \mathbf{V}_1 - \mu_{ji}(1 + \alpha_{ij})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\hat{\boldsymbol{\sigma}}. \quad (9.2)$$

This result applies for both $i = j$ and $i \neq j$.

Let us start with the collision frequency ν defined by (8.7). Use of the identity (9.2) in (8.7) gives

$$\nu = \frac{m_1}{dn_1 T_1} B_3 \sigma_{12}^{d-1} \mu_{21}(1 + \alpha_{12}) \int d\mathbf{V}_1 \int d\mathbf{V}_2 g_{12}[f_{1,M}(\mathbf{V}_1) f_2^{(0)}(\mathbf{V}_2)(\mathbf{V}_1 \cdot \mathbf{g}_{12})$$
$$- \delta\gamma f_1^{(0)}(\mathbf{V}_1) f_{2,M}(\mathbf{V}_2)(\mathbf{V}_2 \cdot \mathbf{g}_{12})], \quad (9.3)$$

where use has been made of the result

$$\int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})^k \hat{\boldsymbol{\sigma}} = B_{k+1} g_{12}^{k-1} \mathbf{g}_{12}, \quad (9.4)$$

with [54]

$$B_k \equiv \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{g}}_{12})^k = \pi^{(d-1)/2} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+d}{2})}. \quad (9.5)$$

Substitution of the Maxwellian approximation (3.11) for $f_i^{(0)}$ gives

$$\nu = \frac{2}{d} \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+3}{2})} \nu_0 (1 + \alpha_{12}) \pi^{-d} (\theta_1 \theta_2)^{d/2} \int d\mathbf{c}_1$$
$$\times \int d\mathbf{c}_2 \, y e^{-(\theta_1 c_1^2 + \theta_2 c_2^2)} [x_2 \gamma_1^{-1}(\mathbf{c}_1 \cdot \mathbf{y}) - x_1 \gamma_2^{-1}(\mathbf{c}_2 \cdot \mathbf{y})], \quad (9.6)$$

where $\mathbf{c}_i \equiv \mathbf{V}_i/v_0$ and $\mathbf{y} \equiv \mathbf{c}_1 - \mathbf{c}_2$. The integral can be performed by the change of variables $\{\mathbf{c}_1, \mathbf{c}_2\} \to \{\mathbf{y}, \mathbf{z}\}$, where $\mathbf{z} \equiv \theta_1 \mathbf{c}_1 + \theta_2 \mathbf{c}_2$ and the Jacobian is $(\theta_1 + \theta_2)^{-d}$. With this change, (9.6) becomes

$$\nu = \frac{2}{d} \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+3}{2})} \nu_0 (1 + \alpha_{12}) \pi^{-d} (\theta_1 \theta_2)^{(d+1)/2} (\theta_1 + \theta_2)^{-(1+d)} (x_2 \mu_{21} + x_1 \mu_{12})$$
$$\times \int d\mathbf{y} \int d\mathbf{z} \, y^3 e^{-(ay^2 + bz^2)}, \quad (9.7)$$

where $a \equiv \theta_1 \theta_2 (\theta_1 + \theta_2)^{-1}$ and $b \equiv (\theta_1 + \theta_2)^{-1}$. The integral (9.7) can be easily computed and one directly gets the result (4.21) given in the text for the reduced collision frequency $\nu^* = \nu/v_0$.

The collision frequencies τ_{ij} defined by (8.30) and (8.31) involve collision integrals of the form

$$\int d\mathbf{v}_1 \mathbf{V}_1 \mathbf{V}_1 J_{ij}[f_i, f_j] = \sigma_{ij}^{d-1} \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_i(\mathbf{V}_1) f_j(\mathbf{V}_2)$$

$$\times \int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})[\mathbf{V}_1''\mathbf{V}_1'' - \mathbf{V}_1\mathbf{V}_1], \qquad (9.8)$$

where the identity (9.1) has been used. The scattering rule (9.2) gives

$$\mathbf{V}_1''\mathbf{V}_1'' - \mathbf{V}_1\mathbf{V}_1 = -\mu_{ji}(1+\alpha_{ij})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})[\mathbf{G}_{ij}\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}}\mathbf{G}_{ij} + \mu_{ji}(\mathbf{g}_{12}\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}}\mathbf{g}_{12})$$

$$- \mu_{ji}(1+\alpha_{ij})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\hat{\boldsymbol{\sigma}}\,\hat{\boldsymbol{\sigma}}], \qquad (9.9)$$

where $\mathbf{G}_{ij} = \mu_{ij}\mathbf{V}_1 + \mu_{ji}\mathbf{V}_2$. Substitution of (9.9) into (9.8) allows the angular integral to be performed with the result

$$\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})[\mathbf{V}_1''\mathbf{V}_1'' - \mathbf{V}_1\mathbf{V}_1]$$

$$= -B_3 m_i \mu_{ji}(1+\alpha_{ij})\Bigg[g_{12}(\mathbf{G}_{ij}\mathbf{g}_{12} + \mathbf{g}_{12}\mathbf{G}_{ij})$$

$$+ 3\frac{\mu_{ji}}{d+3}(1+\frac{2d}{3}-\alpha_{ij})g_{12}\mathbf{g}_{12}\mathbf{g}_{12} - \frac{\mu_{ji}}{d+3}(1+\alpha_{ij})g^3\mathbb{1}\Bigg]. \qquad (9.10)$$

Using (9.10) the integrals defining τ_{ij} can be calculated by the same mathematical steps as those made before for ν. After a lengthy calculation, one gets

$$\int d\mathbf{v}_1 R_{1,k\ell} J_{12}[f_1^{(0)}, f_{2,M} R_{2,k\ell}] = -\frac{\pi^{(d-1)/2}}{2d\Gamma(\frac{d}{2})} m_1 m_2 n_1 n_2 \mu_{21}(1+\alpha_{12})\sigma_{12}^{d-1} v_0^5 (\theta_1\theta_2)^{-1/2}$$

$$\times \Bigg\{ 2(d+3)(d-1)(\mu_{12}\theta_2 - \mu_{21}\theta_1)\theta_2^{-2}(\theta_1+\theta_2)^{-1/2}$$

$$+ 3(d-1)\mu_{21}\left(1+\frac{2d}{3}-\alpha_{12}\right)\theta_2^{-2}(\theta_1+\theta_2)^{1/2}$$

$$- [2d(d+1)-4]\theta_2^{-1}(\theta_1+\theta_2)^{-1/2}\Bigg\}, \qquad (9.11)$$

$$\int d\mathbf{v}_1 R_{1,k\ell} J_{12}[f_{1,M} R_{1,k\ell}, f_2^{(0)}] = -\frac{\pi^{(d-1)/2}}{2d\Gamma(\frac{d}{2})} m_1 m_2 n_1 n_2 \mu_{21}(1+\alpha_{12})\sigma_{12}^{d-1} v_0^5 (\theta_1\theta_2)^{-1/2}$$

$$\times \Bigg\{ 2(d+3)(d-1)(\mu_{12}\theta_2 - \mu_{21}\theta_1)\theta_1^{-2}(\theta_1+\theta_2)^{-1/2}$$

$$+ 3(d-1)\mu_{21}\left(1+\frac{2d}{3}-\alpha_{12}\right)\theta_1^{-2}(\theta_1+\theta_2)^{1/2}$$

$$+ [2d(d+1)-4]\theta_1^{-1}(\theta_1+\theta_2)^{-1/2}\Bigg\}, \qquad (9.12)$$

$$\int d\mathbf{v}_1 R_{1,k\ell}\{J_{11}[f_1^{(0)}, f_{1,M} R_{1,k\ell}] + J_{11}[f_{1,M} R_{1,k\ell}, f_1^{(0)}]\}$$

$$= -\frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} m_1^2 n_1^2 (1 + \alpha_{11})\sigma_1^{d-1}(T_1/m_1)^{5/2}$$

$$\times \frac{6(d-1)}{d}\left(1 + \frac{2d}{3} - \alpha_{11}\right). \tag{9.13}$$

The corresponding expressions for τ_{ij} can be easily inferred from (9.11–9.13).

The collision frequencies ν_{ij} and ω_{ij} that determine the heat flux are defined by (8.30), (8.31), and (8.32), respectively. To evaluate these collision integrals, one needs the partial results

$$\mathbf{S}_i(\mathbf{V}_1'') - \mathbf{S}_i(\mathbf{V}_1) = \frac{m_i}{2}(1 + \alpha_{ij})\mu_{ji}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\bigg\{\bigg[(1 - \alpha_{ij}^2)\mu_{ji}^2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})^2 - G_{ij}^2 - \mu_{ji}^2 g_{12}^2$$

$$- 2\mu_{ji}(\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) + 2(1 + \alpha_{ij})\mu_{ji}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{G}_{ij}) + (d+2)\frac{T_i}{m_i}\bigg]\hat{\boldsymbol{\sigma}}$$

$$- [(1 - \alpha_{ij})\mu_{ji}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) + 2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{G}_{ij})]\mathbf{G}_{ij}$$

$$- \mu_{ji}[(1 - \alpha_{ij})\mu_{ji}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) + 2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{G}_{ij})]\mathbf{g}_{12}\bigg\}, \tag{9.14}$$

$$\int d\hat{\boldsymbol{\sigma}}\,\Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})[\mathbf{S}_i(\mathbf{V}_1'') - \mathbf{S}_i(\mathbf{V}_1)]$$

$$= -\frac{m_i}{2}\frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+3}{2})}(1 + \alpha_{ij})\mu_{ji}$$

$$\times \bigg\{\bigg[g_{12}G_{ij}^2 + \mu_{ji}^2\frac{4\alpha_{ij}^2 - (d+3)\alpha_{ij} + 2(d+1)}{d+3}g_{12}^3$$

$$- 2\mu_{ji}\frac{3\alpha_{ij} - 2d - 3}{d+3}g_{12}(\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) - (d+2)\frac{T_i}{m_i}g_{12}\bigg]\mathbf{g}_{12}$$

$$+ \bigg[2g_{12}(\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) - \mu_{ji}\frac{(d+5)\alpha_{ij} - d - 1}{d+3}g_{12}^3\bigg]\mathbf{G}_{ij}\bigg\}. \tag{9.15}$$

The integrals ω_{ij} and ν_{ij} can be explicitly evaluated by using (9.15) and the same mathematical steps as before. After a lengthy algebra, one gets

$$\omega_{12} = \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})}\frac{2}{d\sqrt{2}}\left(\frac{\sigma_1}{\sigma_{12}}\right)^{d-1} x_1 \theta_1^{-1/2}(1 - \alpha_{11}^2)$$

$$+ \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})}\frac{2}{d(d+2)} x_1 \mu_{21}(1 + \alpha_{12})(\theta_1 + \theta_2)^{-1/2}\theta_1^{1/2}\theta_2^{-3/2}\left(\frac{x_2}{x_1}A - \gamma B\right), \tag{9.16}$$

$$\nu_{11} = \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})}\frac{8}{d(d+2)}\left(\frac{\sigma_1}{\sigma_{12}}\right)^{d-1} x_1(2\theta_1)^{-1/2}(1 + \alpha_{11})$$

$$\times \left[\frac{d-1}{2} + \frac{3}{16}(d+8)(1-\alpha_{11}) \right]$$

$$+ \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} \frac{1}{d(d+2)} x_2 \mu_{21}(1+\alpha_{12}) \left(\frac{\theta_1}{\theta_2(\theta_1+\theta_2)} \right)^{3/2}$$

$$\times \left[E - (d+2)\frac{\theta_1+\theta_2}{\theta_1} A \right], \tag{9.17}$$

$$\nu_{12} = -\frac{\pi^{(d-1)/2}}{\Gamma(\frac{d}{2})} \frac{1}{d(d+2)} x_2 \frac{\mu_{21}^2}{\mu_{12}}(1+\alpha_{12}) \left(\frac{\theta_1}{\theta_2(\theta_1+\theta_2)} \right)^{3/2} \left[F + (d+2)\frac{\theta_1+\theta_2}{\theta_2} B \right]. \tag{9.18}$$

In the above equations we have introduced the quantities[2]

$$A = (d+2)(2\beta_{12}+\theta_2)$$
$$+ \mu_{21}(\theta_1+\theta_2)\{(d+2)(1-\alpha_{12}) - [(11+d)\alpha_{12} - 5d - 7]\beta_{12}\theta_1^{-1}\}$$
$$+ 3(d+3)\beta_{12}^2\theta_1^{-1} + 2\mu_{21}^2\left(2\alpha_{12}^2 - \frac{d+3}{2}\alpha_{12} + d + 1\right)\theta_1^{-1}(\theta_1+\theta_2)^2$$
$$- (d+2)\theta_2\theta_1^{-1}(\theta_1+\theta_2), \tag{9.19}$$

$$B = (d+2)(2\beta_{12}-\theta_1)$$
$$+ \mu_{21}(\theta_1+\theta_2)\{(d+2)(1-\alpha_{12}) + [(11+d)\alpha_{12} - 5d - 7]\beta_{12}\theta_2^{-1}\}$$
$$- 3(d+3)\beta_{12}^2\theta_2^{-1} - 2\mu_{21}^2\left(2\alpha_{12}^2 - \frac{d+3}{2}\alpha_{12} + d + 1\right)\theta_2^{-1}(\theta_1+\theta_2)^2$$
$$+ (d+2)(\theta_1+\theta_2), \tag{9.20}$$

$$E = 2\mu_{21}^2\theta_1^{-2}(\theta_1+\theta_2)^2\left(2\alpha_{12}^2 - \frac{d+3}{2}\alpha_{12} + d + 1\right)[(d+2)\theta_1 + (d+5)\theta_2]$$
$$- \mu_{21}(\theta_1+\theta_2)\{\beta_{12}\theta_1^{-2}[(d+2)\theta_1 + (d+5)\theta_2][(11+d)\alpha_{12} - 5d - 7]$$
$$- \theta_2\theta_1^{-1}[20 + d(15 - 7\alpha_{12}) + d^2(1-\alpha_{12}) - 28\alpha_{12}] - (d+2)^2(1-\alpha_{12})\}$$
$$+ 3(d+3)\beta_{12}^2\theta_1^{-2}[(d+2)\theta_1 + (d+5)\theta_2]$$
$$+ 2\beta_{12}\theta_1^{-1}[(d+2)^2\theta_1 + (24 + 11d + d^2)\theta_2]$$
$$+ (d+2)\theta_2\theta_1^{-1}[(d+8)\theta_1 + (d+3)\theta_2]$$
$$- (d+2)(\theta_1+\theta_2)\theta_1^{-2}\theta_2[(d+2)\theta_1 + (d+3)\theta_2], \tag{9.21}$$

$$F = 2\mu_{21}^2\theta_2^{-2}(\theta_1+\theta_2)^2\left(2\alpha_{12}^2 - \frac{d+3}{2}\alpha_{12} + d + 1\right)[(d+5)\theta_1 + (d+2)\theta_2]$$
$$- \mu_{21}(\theta_1+\theta_2)\{\beta_{12}\theta_2^{-2}[(d+5)\theta_1 + (d+2)\theta_2][(11+d)\alpha_{12} - 5d - 7]$$

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[2]See footnote 1.

$$+ \theta_1 \theta_2^{-1} [20 + d(15 - 7\alpha_{12}) + d^2(1 - \alpha_{12}) - 28\alpha_{12}] + (d+2)^2(1 - \alpha_{12})\}$$
$$+ 3(d+3)\beta_{12}^2 \theta_2^{-2} [(d+5)\theta_1 + (d+2)\theta_2]$$
$$- 2\beta_{12}\theta_2^{-1}[(24 + 11d + d^2)\theta_1 + (d+2)^2\theta_2]$$
$$+ (d+2)\theta_1 \theta_2^{-1}[(d+3)\theta_1 + (d+8)\theta_2]$$
$$- (d+2)(\theta_1 + \theta_2)\theta_2^{-1}[(d+3)\theta_1 + (d+2)\theta_2]. \tag{9.22}$$

Here, $\beta_{12} = \mu_{12}\theta_2 - \mu_{21}\theta_1$. From (9.16–9.22), one easily gets the expressions for ω_{21}, ν_{22} and ν_{21} by interchanging $1 \leftrightarrow 2$.

In the case of a three-dimensional system ($d = 3$), all the above results reduce to those previously obtained for hard spheres when one takes Maxwellian distributions for the reference homogeneous cooling state [26, 28].

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